Construction of Harmonic Maps by MinMax Methods

Thesis for Master in Mathematics

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Abstract. Using the MinMax procedure introduced by Palais we construct critical points of the perturbed Dirichlet energy $\int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2$. By passing to the limit when $\sigma \to 0$, we construct a weakly harmonic map from S^3 to S^2. We conjecture that this harmonic map is the Hopf fibration realizing the width of the MinMax problem.

Contents

1. Introduction

In this paper we want to study harmonic maps between Riemannian manifolds. They are critical points of the Dirichlet energy (section 2),

$$E(u) = \int_M |\nabla u|^2 dvol_g$$

Variation of such an energy leads us to an Euler-Lagrange equation, which contains nonlinearity due to the geometry of the Riemannian manifolds, and causes difficulties in studying the regularity of the critical points.

However, we do get regularity once we know that the weakly harmonic map is continuous, and (partial) continuity can be achieved when a monotonicity formula is at hand. The following two references for harmonic maps and partial regularity are very well-written, Hèlein [13] and Moser [20]. Also one may refer to the two reports for examples and properties of harmonic maps as of the 70s and 80s during the development of these theories, [6] and [7].

Notice that the minimal value of the Dirichlet energy would be reached by constant maps, so we would resort to a well-built admissible family (section 5) and apply the procedure of MinMax.

One could consider the following family,

$$\mathcal{A}^0 = \Big\{ u \in C^0(\overline{B^4}, W^{1,2}(S^3, S^2)) \Big| \max_{\bar{x} \in \partial B^4} \int_{S^3} |\nabla u(\bar{x}, \cdot)|^2 \leq \frac{1}{2} C_{S^3}, \quad \frac{\overline{u(\bar{x}, \cdot)}}{|\overline{u(\bar{x}, \cdot)}|} \text{ is not null-homotopic from } S^3 \text{ to } S^2 \Big\}$$

Here C_{S^3} is a positive constant that depends on the constant of the Poincarè inequality, defined in section 5, and $\overline{f(\cdot)} = \fint f(x)$ is the notation for mean integral over the domain.

And one would want to find critical points realizing the following positive value,

$$0 < \bar{\beta}(0) = \inf_{u \in \mathcal{A}^0} \sup_{x \in \overline{B^4}} \int_{S^3} |\nabla u|^2$$

But notice that $W^{1,2}(S^3, S^2)$ is not a Banach manifold. Therefore one cannot apply MinMax on it, or talk about the critical points realizing the width.

Hence in this paper we will start with critical points of the perturbed Dirichlet energy,

$$E_\sigma(u) = \int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2$$

in the space $W^{2,2}(S^3, S^2)$, see Lamm [14] or Rivière [24]. The width with respect to this energy is defined as,

$$\beta(\sigma) := \inf_{u \in \mathcal{A}} \max_{x \in \overline{B^4}} E_\sigma(u(x, \cdot)) = \inf_{u \in \mathcal{A}} \max_{x \in \overline{B^4}} \int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2 dvol_g$$

E_σ is Palais-Smale, and the PDE of the critical points will be subcritical and regularity follows.

Theorem (4.1). *For any $\sigma > 0$, the functional $E_\sigma(u)$ for $u \in W^{2,2}(S^3, S^2)$ satisfies the Palais-Smale condition, that is, given $(u_n)_{n \in \mathbb{N}} \in W^{2,2}(S^3, S^2)$ such that,*

$$DE_\sigma(u_n) \to 0, \quad E_\sigma(u_n) \to \alpha(\sigma) \in \mathbb{R}$$

there is a subsequence u_n (without relabeling) that converges to u in the $W^{2,2}(S^3, S^2)$ norm, which is equivalent to the convergence in the Finsler metric. It follows directly from the continuity of $E_\sigma(\cdot)$ and $DE_\sigma(\cdot)$ that $E_\sigma(u) = \alpha(\sigma)$ and $DE_\sigma(u) = 0$

Notice here we have a submanifold, $W^{2,2}(S^3, S^2)$ of the Hilbert space $W^{2,2}(S^3, \mathbb{R}^3)$, then the Finsler structure it has is well-behaved. To be precise, the author obtained the following estimates.

Lemma (2.14). *There is a constant $C > 0$, such that for any $u, v \in W^{2,2}(S^3, S^2)$, the distance generated by the Finsler metric $d(u, v)$ is comparable to the Sobolev distance $\|u - v\| := \|u - v\|_{W^{2,2}}$,*

$$C^{-1} d(u, v) \leq \|u - v\| \leq C d(u, v)$$

So that we can see the two distances generate the same topology.

Theorem (4.4). *Critical points of E_σ, satisfying equation (3.1), are smooth.*

For domain with dimension 4, we remark that in Chang-Wang-Yang [5], they have proved a regularity theory for biharmonic maps into a sphere using Morrey estimates, and that will be related to our perturbed term. For dimension higher than 4, there they also developed a monotonicity formula for stationary biharmonic maps for partial regularity.

Now for E_σ, we would adapt the admissible family above as,

$$\mathcal{A} = \Big\{ u \in C^0(\overline{B^4}, W^{2,2}(S^3, S^2)) \Big| \max_{\bar{x} \in \partial B^4} \int_{S^3} |\nabla u(\bar{x}, \cdot)|^2 + \sigma_0^2 |\Delta u(\bar{x}, \cdot)|^2 \leq \frac{1}{2} C_{S^3}, \\ \frac{\overline{u(\bar{x}, \cdot)}}{|\overline{u(\bar{x}, \cdot)}|} \text{ is not null-homotopic from } S^3 \text{ to } S^2 \Big\}$$

for some $\sigma_0 > 0$ to be decided in section 5.

After showing that this is indeed well-defined for the MinMax procedure and the width with respect to E_σ on the family $\mathcal{A}$ is positive, we would like to pass to the limit as $\sigma \to 0$.

However, we don't know if the limit will realize the width $\beta(0)$. One difficulty is that we only have $W^{1,2}$ bound on the perturbed critical points independent of σ, and we don't know if the following is true,

$$\sigma^2 \int_{S^3} |\Delta u|^2 \to 0 \quad ?$$

However, this could be solved by using an "entropy condition" argument (section 6) like in Struwe's paper [29].

Theorem (6.3). *There is a sequence $\sigma_n \to 0$ and $u_n \in W^{2,2}(S^3, S^2)$, such that*

$$\|DE_{\sigma_n}(u_n)\| = 0, \quad E_{\sigma_n}(u_n) = \beta(\sigma_n), \quad (\partial_\sigma E_\sigma(u_n))_{\sigma=\sigma_n} = o(\frac{1}{\sigma_n \log(\frac{1}{\sigma_n})})$$

Using the special structure of the sphere as our target space, we can actually solve the Euler-Lagrange equation for the limit directly and confirm that it is weakly harmonic.

Theorem (7.1). *Given the admissible family $\mathcal{A}$ built in section 5, assume $(u_n)_{n\in\mathbb{N}} \in W^{2,2}(S^3, S^2)$ are critical points of E_{σ_n} for σ_n as in section 6, such that*

$$\lim_{\sigma_n \to 0} \sigma_n^2 \int_{S^3} |\Delta u_n|^2 \to 0$$

also,

$$E_{\sigma_n}(u_n) = \inf_{u\in\mathcal{A}} \sup_{x\in\bar{B}^4} E_{\sigma_n}(u) = \beta_{\sigma_n} \to \beta_0$$

we can find a subsequence converging weakly in $W^{1,2}$ to a weakly harmonic map u from S^3 to S^2.

Another difficulty would be that we need strong $W^{1,2}$ convergence to pass on the width, and this is yet to be solved (see section 8). We remark that in Lamm's paper [14], this is indeed true in dimension 2, and in fact the convergence is arbitrarily smooth away from finitely many points. To solve this, we propose that one can look at the monotonicity formula for critical points of E_σ. Furthermore, we conjectured that in fact our nontrivial limit has Morse Index no more than 4 and thus it is the Hopf map as below, as conjectured by Tristan Rivière,

$$h : \mathbb{R}^4 \supset S^3 \to S^2 \subset \mathbb{R}^3, \quad h(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\overline{z_2})$$

by identifying $\mathbb{R}^2$ with $\mathbb{C}$.

Also, this solves another conjecture given by Rivière. It is proved in his paper in 1998 [25] that the above Hopf fibration minimizes the 3-energy among the symmetric fibrations from S^3 to S^2. If the width $\beta(0)$ is realized by the Hopf map, then we have,

$$\left|S^3\right|^{\frac{1}{3}} \left(\int_{S^3} |\nabla h|^3\right)^{\frac{2}{3}} \overset{?}{=} \beta(0) \leq \sup_{a\in\overline{B^4}} E(u\circ\varphi_a) \leq \left|S^3\right|^{\frac{1}{3}} \left(\int_{S^3} |\nabla u|^3\right)^{\frac{2}{3}}$$

using the symmetry of the Hopf map, the conformal invariance of the 3-energy and Hölder inequality. Here φ_a is a conformal diffeomorphism from S^3 to S^3. The question mark over the first equality indicates the difficult parts left open in this work.

That is, the Hopf map minimizes the 3-energy among all $W^{1,3}(S^3, S^2)$ maps that is not null-homotopic. Please refer to the last section for more details.

2. Basic Set-Up: Global Nonlinear Analysis

To study harmonic maps between two closed (compact without boundary) smooth Riemannian manifolds $M \to N$, we first need to define weakly differentiable functions on a manifold and Sobolev spaces in this setting. The basic reference are Adams and Fournier [1] (weakly differentiable functions on domains in Euclidean spaces), Hebey's book [12] (functions between Riemannian manifolds).

Firstly, by Nash's embedding theorem, N embeds isometrically into $\mathbb{R}^q (q \in \mathbb{N})$. We write (M^m, g) and (N^n, h) for the metrics and the dimensions.

Definition 2.1. *The following is the vector-valued weakly differentiable functions,*

$$W^{k,p}(M, \mathbb{R}^q) := \overline{C^{k,p}(M, \mathbb{R}^q)}^{\|\cdot\|_{W^{k,p}}}$$

$$C^{k,p}(M, \mathbb{R}^q) := \{u : M \to \mathbb{R}^q, u \in C^\infty, \forall j = 0, ..., k, \nabla^j u \in L^p(M),$$
$$i.e. \|u\|_{W^{k,p}} = \sum_{j=0}^{k} \int_M |\nabla^j u|^p dvol_g < \infty\}$$

where ∇ is the Levi-Civita connection associated to g.

$$W^{k,p}(M, N) := \{u \in W^{k,p}(M, \mathbb{R}^q), u(x) \in N \, a.e.\}.$$

We know that for $p = 2$, $W^{k,2}(M, \mathbb{R}^q)$ is a Hilbert space. For general target manifold, before studying the functional defined on them, let's first give the definition of a Banach manifold and submanifold for Hausdorff topological spaces. For a complete treatment please refer to Lang's *Fundamentals of Differential Geometry* [15].

Definition 2.2. *Assume $\mathcal{M}$ is a Hausdorff topological space, we say that $\mathcal{M}$ is a Banach manifold, if for any $x \in \mathcal{M}$, there is an open neighborhood and a homeomorphism (U, ϕ) into E, some Banach space.*

We call the collection of the maps and neighborhoods that cover $\mathcal{M}$ an atlas $\mathcal{U}$ for the Banach manifold, and each $(U_i, \phi_i) \in \mathcal{U}$ is a chart. If the intersection $U_i \cap U_j$ is not empty, then we have the transition map $\phi_{ij} = \phi_i \circ \phi_j^{-1}$, and also the respective Banach spaces E_i, E_j must be isomorphic as topological vector spaces. When the transition maps are C^m for $m = 0, 1, 2, ..., \infty$, we then call the manifold a C^m Banach manifold.

The definition for submanifold requires more care than that of a finite dimensional manifold.

Definition 2.3. *A subset $\mathcal{M}_0$ of $\mathcal{M}$ is a (C^m) submanifold if for any $x \in \mathcal{M}_0$, there is a chart in $\mathcal{M}$, say (U, ϕ) into a Banach space E, and a direct sum decomposition $E = E_1 + E_2$ such that,*

$$\phi(\mathcal{M}_0 \cup U) = \phi(U) \cup E_1$$

Furthermore, if $\mathcal{M}_0$ is closed in $\mathcal{M}$, then it is called a closed submanifold.

A (Banach) submanifold satisfy the following universal property,

Lemma 2.4 (Lang, [15])**.** *Assume $\mathcal{M}_0$ is a (Banach) submanifold of $\mathcal{M}$, then any C^m map $f : N \to \mathcal{M}$ whose image lies in $\mathcal{M}_0$ is also a C^m map into $\mathcal{M}_0$.*

Theorem 2.5. *$W^{k,p}(M, N)$ is a Banach manifold for $kp > m$.*

Proof. The proof we give here follows from the notes written by Tristan Rivière [26]. Notice that by Sobolev embedding $W^{k,p}(M, N) \hookrightarrow C^0(M, N)$, for any $\epsilon > 0$, we have if $v \in B_\epsilon(u)$ with respect to the Sobolev distance, then $\sup_{x \in M} |u(x) - v(x)| \leq C_0 \epsilon$, with C_0 the constant of embedding. Since N is compact, we may assume there is a small constant $\delta > 0$, such that the geodesic ball $B_\delta(z)$ for any point z on N, is strictly convex and the smooth exponential map is well-defined in the unit ball $B_1 \in T_p N$ for all $p \in S^2$. Now the geodesic distance of $u(x)$ and $v(x)$ on N is comparable to the Euclidean distance $|u(x) - v(x)|$ (by a constant uniformly on N). Thus we can define the maps for any $u \in W^{k,p}(M, N)$,

$$l_u : B_\epsilon(u) \to E_u, \;\; l_u(v)(x) := exp^{-1}_{u(x)}(v(x))$$

$$E_u := \{w(x) \in W^{k,p}(M, \mathbb{R}^q), w(x) \in T_{u(x)} N\}$$

Now we have these coordinate maps,

$$l_v \circ (l_u)^{-1} : l_u(B_\epsilon(u) \cap B_\epsilon(v)) \to l_v(B_\epsilon(u) \cap B_\epsilon(v))$$

This is a map from a subset of the unit ball $B_1 \in E_u$ to E_v. One can see that this map is smooth, for example,

$$L(w(x)) := l_v \circ (l_u)^{-1}(w(x))$$

$$\begin{aligned} L(w(x)+h(x)) - L(w(x)) &= exp^{-1}_{v(x)} \circ exp_{u(x)}(w(x)+h(x)) - exp^{-1}_{v(x)} \circ exp_{u(x)}(w(x)) \\ &= T(u(x), w(x)+h(x)) - T(u(x), w(x)) \\ &= D_2T(u(x), w(x))h(x) + \int_0^1 D_2^2 T[h,h](u, w+th)dt \end{aligned}$$

under Taylor expansion. Now one can apply Minkowski's inequality,

$$\begin{aligned} &\|L(w(x)+h(x)) - L(w(x)) - D_2T(u(x), w(x))h(x)\|_{W^{k,p}} \\ &= \left\| \int_0^1 D_2^2 T[h,h](u, w+th)dt \right\|_{W^{k,p}} \\ &\le \int_0^1 \left\| D_2^2 T[h,h](u, w+th) \right\|_{W^{k,p}} dt \\ &\le \int_0^1 \left\| D_2^2 T(u, w+th) \right\|_{W^{k,p}} \|h\|^2_{W^{k,p}} \\ &\le C(\|u\|_{W^{k,p}}, \|w\|_{W^{k,p}}) \|h\|^2_{W^{k,p}} \end{aligned}$$

For the last inequality we used that the Sobolev space $W^{k,p}$ for $kp > m$, with m the dimension of the domain manifold, is a Banach Algebra and that composition with a smooth function, i.e. $\Phi(u_1, u_2, ..., u_s)$ with $\Phi(\cdot)$ smooth– this is a continuous map from $\oplus W^{k,p}$ to $W^{k,p}$. One may prove this using chain rule or refer to Lemma 9.9 in Palais' lecture notes [23]. This gives us C^1 (smoothness is similar). □

The above theorem justifies why a higher order perturbation of the Dirichlet energy is needed for building a harmonic map in $W^{1,2}(S^3, S^2)$, whose definition we give now. The basic reference is Roger Moser's book [20].

Definition 2.6. *A map $u \in W^{1,2}(M, N)$ is a (weakly) harmonic map, if it is a critical point of the Dirichlet energy E with respect to compactly supported variations on the target manifold.*

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 dvol_g$$

That is, for any $\varphi \in C_c^\infty(M, \mathbb{R}^q)$, writing π_N for the nearest point projection of $N \hookrightarrow \mathbb{R}^q$, then

$$\left.\frac{d}{dt}\right|_{t=0} (E(\pi_N(u + t\varphi))) = 0$$

Notice that here we write the Dirichlet energy using the isometric embedding of N into $\mathbb{R}^q$. In section 3 we will derive the Euler Lagrange equation given by this definition.

In this paper, we first work on the Banach Manifold $W^{2,2}(S^3, S^2)$, and in order to have coerciveness, we add a viscosity term to get the following functional.

$$E_\sigma(u) = \frac{1}{2}\int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2 dvol_g$$

where the Laplace-Beltrami operator is defined in local coordinate as

$$\Delta u = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} g^{ij} \partial_j u)$$

One may check this definition is independent of the chosen coordinate system.

Lemma 2.7. *E_σ is a C^1 functional on $W^{2,2}(S^3, S^2)$.*

Proof. Firstly, the functional E_σ is C^1 on the Hilbert space $W^{2,2}(S^3, \mathbb{R}^3)$. Indeed, since the integrand is quadratic,

$$E_\sigma(u+v) - E_\sigma(u) - \int_{S^3} \langle \nabla u, \nabla v \rangle + \sigma^2 \Delta u \cdot \Delta v = E_\sigma(v) = O(\|v\|^2_{W^{2,2}})$$

Now the map $W^{2,2}(S^3, S^2) \hookrightarrow W^{2,2}(S^3, \mathbb{R}^3)$ is smooth due to the lemma below. □

Lemma 2.8. *The map $W^{2,2}(S^3, S^2) \hookrightarrow W^{2,2}(S^3, \mathbb{R}^3)$ is smooth and since $W^{2,2}(S^3, S^2)$ is closed in $W^{2,2}(S^3, \mathbb{R}^3)$, it's a closed submanifold.*

Proof. One may work in local charts for $W^{2,2}(S^3, S^2)$ as in Theorem 2.5. Since our target is S^2, we can also use the nearest point projection $\frac{\cdot}{|\cdot|}$ to set up a local chart for $W^{2,2}(S^3, S^2)$. Let u, $v \in B_\epsilon(u)$, $w \in E_u$ as in Theorem 2.5,

$$E_u \ni w \to \frac{u+w}{|u+w|} \in B_\epsilon(u)$$

is a smooth local chart for $W^{2,2}(S^3, S^2)$. The proof is similar to Theorem 2.5. The claim now follows. □

The following definition of Finsler manifold follows from Palais [22].

Definition 2.9. *A $C^r (r > 0)$ Finsler manifold M is a regular C^r Banach manifold with a Finsler structure on the tangent bundle TM, that is, a continuous function $\|\cdot\|$ on TM such that the following is true,*

(1) If restricted to each fiber $T_x M$, $\|\cdot\|_x$ is equivalent to the norm on the tangent space;

(2) For any $\epsilon > 0$, $x_0 \in M$, there is a neighborhood $B_r(x_0)$, writing the tangent bundle as $\phi : TM \to M$, and for each local trivialization in some atlas of the tangent bundle,

$$T : \phi^{-1}(B_r(x_0)) \to B_r(x_0) \times T_{x_0} M$$

the norm is comparable as

$$(1-\epsilon)\|v\|_x \leq \|T(x,v)\|_{x_0} \leq (1+\epsilon)\|v\|_x, \quad \textit{for any } (x,v) \in \phi^{-1}(B_r(x_0))$$

For instance, Riemannian manifolds with a C^1 metric is a special example of a C^1 Finsler manifold.

We also need to build a Finsler structure on $W^{2,2}(S^3, S^2)$ before we can apply the Min-Max procedure as in Palais' paper [22]. Notice that $W^{2,2}(S^3, \mathbb{R}^3)$ is a Hilbert space and we already have a norm on the tangent space.

Definition 2.10. *We define the Finsler structure on $W^{2,2}(S^3, S^2)$,*

$$\|w\|_u := \|w\|_{W^{2,2}(S^3,\mathbb{R}^3)}, \quad w \in E_u := \{v \in W^{2,2}(S^3, \mathbb{R}^3), v(x) \in T_{u(x)}S^2\}$$

We first make some remarks before proving that this is a well-defined Finsler structure. Notice that one can apply the Sobolev embedding $W^{2,2}(S^3, \mathbb{R}^3) \hookrightarrow C^0(S^3, \mathbb{R}^3)$. Thus a Cauchy sequence in $W^{2,2}(S^3, S^2)$ also converges uniformly, and the compactness of S^2 implies that the limit also lies in $W^{2,2}(S^3, S^2)$. Hence as we mentioned before, $W^{2,2}(S^3, S^2)$ is a closed subset of the Hilbert space $W^{2,2}(S^3, \mathbb{R}^3)$ and a closed submanifold. In particular, it's a complete metric space. Regularity (and normality) follows directly. Also, knowing that $W^{2,2}(S^3, S^2)$ is a complete Hilbert manifold is already enough for us to apply the MinMax procedure (there one can use the gradient flows, see Palais [21]), however, we can also use this property to directly construct a Finsler structure on the tangent bundle.

Definition 2.11. *Given a Finsler structure $\|\cdot\|_p (p \in M)$ on a regular (or normal) Banach manifold M, we define the distance function,*

$$d(p,q) := \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt$$

where the infimum is taken over all piecewise C^1 paths from p to q.

If the manifold is not path connected and when we are interested in the length of d only when for example $d < 1$, we can truncate the above distance function at 1.

Remark 2.12. Notice that one can prove that the above function is indeed a metric, which is the part where the regularity (or normality) of the Banach manifold is used. This is not totally trivial, see Palais [22] for a complete proof. The proof of the fact that the topology generated by the Finsler distance is the same as the topology of the manifold, is contained in Lemma 2.14.

Lemma 2.13. *$W^{2,2}(S^3, S^2)$ is a complete smooth Finsler manifold.*

Proof. We first show that the Finsler norm is well defined. We use the same notation as in Theorem 2.5. Let $u, v \in W^{2,2}(S^3, S^2)$ and $||u - v||_{W^{2,2}} \leq \epsilon$. We build the trivialization via the map,

$$E_u \ni w \to \frac{u + w}{|u + w|} \in B_\epsilon(u)$$

Then one can calculate that the inverse is given as,

$$l_u(v) : B_\epsilon(u) \to E_u, \quad l_u(v) = w = \frac{v}{u \cdot v} - u \tag{2.1}$$

Consider (v, q) with $q \in E_v$. We take a curve $\gamma(t, x)$ with $\gamma(0, x) = v$ and $\gamma'(0, x) = q$. Composing it we get $\bar{\gamma}(t, x) = l_u \circ \gamma(t, x)$, and $\bar{\gamma}'(0, x) = dl_u(q)$, thus,

$$dl_u(q) = \bar{\gamma}'(0, x) = (\frac{\bar{\gamma}(t, x)}{\bar{\gamma}(t, x) \cdot u(x)} - u)' = \frac{(v \cdot u)q - (q \cdot u)v}{(v \cdot u)^2}$$

Now we compare $||q||_{W^{2,2}(S^3, v^{-1}TS^2)}$ and $||dl_u(q)||_{W^{2,2}(S^3, u^{-1}TS^2)}$,

$$\begin{aligned}
\left\| \frac{q \cdot u}{(v \cdot u)^2} v \right\|_{W^{2,2}} &= \left\| \frac{q \cdot (u - v)}{(v \cdot u)^2} v \right\|_{W^{2,2}} \\
&\leq C \sum_k \left\| \frac{v^k}{(v \cdot u)^2} q \right\|_{W^{2,2}} ||u - v||_{W^{2,2}} \\
&\leq C\epsilon \left\| \frac{v}{(v \cdot u)^2} \right\|_{W^{2,2}} ||q||_{W^{2,2}} \\
&\leq \epsilon C_u \; ||q||_{W^{2,2}}
\end{aligned}$$

The other part has,

$$\begin{aligned}
\left\| \frac{v \cdot u}{(v \cdot u)^2} q - q \right\|_{W^{2,2}} &= \left\| \frac{1 - v \cdot u}{(v \cdot u)} q \right\|_{W^{2,2}} \\
&= \left\| \frac{|v - u|^2}{(v \cdot u)} \frac{q}{2} \right\|_{W^{2,2}} \\
&\leq \epsilon^2 C_u ||q||_{W^{2,2}}
\end{aligned}$$

Together we have,

$$||dl_u(q) - q||_{W^{2,2}(S^3, \mathbb{R}^3)} \leq C_u(\epsilon + \epsilon^2)||q||_{W^{2,2}(S^3, \mathbb{R}^3)}$$

Thus using the triangle inequality of the norm $|| \cdot ||_{W^{2,2}(S^3, \mathbb{R}^3)}$ and choosing ϵ properly we have shown that this is indeed a Finsler structure.

We still need to check that the space $W^{2,2}(S^3, S^2)$ is complete with respect to the Finsler structure. Here we use Lemma 2.14 below that says that the distance generated by the

Finsler structure bounds the Sobolev distance. Any Cauchy sequence in the Finsler metric is also Cauchy in the Sobolev norm, then since $W^{2,2}(S^3,\mathbb{R}^3)$ is a Hilbert space and S^2 is compact, Sobolev embedding gives us uniform convergence on S^3, which implies the limit lies in $W^{2,2}(S^3,S^2)$. □

Lemma 2.14. *There is a constant $C > 0$, such that for any $u, v \in W^{2,2}(S^3,S^2)$, the distance generated by the Finsler metric $d(u,v)$ is comparable to the Sobolev distance $\|u-v\|$,*

$$C^{-1}d(u,v) \leq \|u-v\| \leq Cd(u,v)$$

So that we can see the two distance generate the same topology.

Proof. We work in the same set up as in Theorem 2.5, with $B_\epsilon(u)$, E_u, l_u defined in the same way, we can either take the trivialization induced by the nearest point projection map or the exponential map.

For any path $\gamma(t,x)$ with $\gamma(0,x) = u$ and $\gamma(1,x) = v$, let $\bar{\gamma} := l_u \circ \gamma(t,x)$, then we have the following estimates as of Palais [22],

(2.2)
$$\int_0^1 \|\gamma'(t)\|_{\gamma(t)}dt \geq (1-\varepsilon)\int_0^1 \|\bar{\gamma}'(t)\|_u dt \geq (1-\varepsilon)\left\|\int_0^1 \bar{\gamma}'(t)dt\right\|_u = (1-\varepsilon)\|l_u(u)-l_u(v)\|_u$$

We define the map

$$\Phi(x_1,x_2) := \frac{x_1+x_2}{|x_1+x_2|}$$

which is smooth away from $\{(x_1,x_2), x_1+x_2=0\}$, and notice $l_u^{-1}(w) = \Phi(u(x),w(x))$.

Recall our map in equation 2.1, since $l_u(u) = 0 \in E_u$, we have for fixed u and any $v \in B_\epsilon(u)$, $(u-v)(x) = (\Phi(u(x),0) - \Phi(u(x),w(x)))$, where $0 \in E_u = W^{2,2}(S^3,u^{-2}TS^2)$ and $w = l_u(v) \in E_u$. Then again applying chain rule we have,

$$\|u-v\|_{W^{2,2}} = \|\Phi(u(x),0) - \Phi(u(x),w(x))\|_{W^{2,2}} \leq C_u\|w\|_{W^{2,2}} = C_u\|l_u(u)-l_u(v)\|_{W^{2,2}}$$

Plug the above estimates into equation 2.2 and taking the infimum we obtain the inequality on the right.

Notice that using the same technique we can achieve the following bound for the inequality on the left.

Again writing $w = l_u(v) \in E_u$, then we define the curve $\gamma_0(t,x) = t(w(x))$ in E_u and compose it to get $\gamma(t,x) = l_u^{-1} \circ \gamma_0$,

$$\int_0^1 \|\gamma'(t,x)\|_{\gamma(t,x)}dt \leq (1+\varepsilon)\int_0^1 \|w\|_u dt = (1+\varepsilon)\|w\|_u$$

Hence $d(u, v) \leq 2\|w\|_u$. Then once can apply the same argument above to get the inequality on the left.

Alternatively, we can also work in the space $W^{2,2}(S^3, \mathbb{R}^3)$. Assume $u, v \in W^{2,2}(S^3, S^2)$ and $\|u-v\|_{W^{2,2}} \leq \epsilon \leq \frac{1}{2}C_0^{-1}$, where C_0 is the constant for the Sobolev Embedding $W^{2,2}(S^3, \mathbb{R}^3)$ into $C^0(S^3, \mathbb{R}^3)$. Then the following is a well-defined C^1 path from u to v within $W^{2,2}(S^3, S^2)$,

$$\gamma(t, x) = \frac{tu(x) + (1-t)v(x)}{|tu(x) + (1-t)v(x)|} = \frac{\bar{\gamma}(t, x)}{|\bar{\gamma}(t, x)|}$$

Notice that clearly $\bar{\gamma}(t, x)$ is smooth with respect to t into $W^{2,2}(S^3, \mathbb{R}^3)$, and the map $\frac{\bar{\gamma}(t,x)}{|\bar{\gamma}(t,x)|}$ is smooth on the open set $B_\epsilon(u) := \{v \in W^{2,2}(S^3, \mathbb{R}^3), \|u - v\|_{W^{2,2}} < \epsilon\}$ of the Hilbert space $W^{2,2}(S^3, \mathbb{R}^3)$, and its image lies in $W^{2,2}(S^3, S^2)$. Since $W^{2,2}(S^3, S^2)$ is a submanifold, by the universal property mentioned before, we know that the map is also smooth into the Banach manifold $W^{2,2}(S^3, S^2)$.

We calculate the length of this path,

$$l = \int_0^1 \|\gamma'(t, x)\|_{W^{2,2}} dt = \int_0^1 \left\| \frac{u - v}{|\bar{\gamma}|} - \frac{\bar{\gamma} \cdot (u - v)}{|\bar{\gamma}|^3} \bar{\gamma} \right\|_{W^{2,2}} dt$$

One may calculate that the integrand can be bounded by $C\|u - v\|_{W^{2,2}}$ independent of t, which gives another proof.

□

3. The Euler-Lagrange Equation

In this section we calculate the Euler-Lagrange equation of the functional E_σ. Notice that the tangent space to $W^{2,2}(S^3, S^2)$ at u depends on the values $u(x) \in S^2$. For convenience we write the orthogonal projection $P_u(v(x))$ for $v(x) \in W^{2,2}(S^3, \mathbb{R}^3)$ as

$$P_u(v(x)) = v(x) - (v(x) \cdot u(x))u(x)$$

Since our domain and target manifolds are smooth, for $v(x) \in W^{2,2}(S^3, \mathbb{R}^3)$, and $u \in W^{2,2}(S^3, S^2)$, $P_u(v(x))$ is in $W^{2,2}(S^3, u^{-1}TS^2)$. We write $\langle , \rangle$ for the product with respect to the metric on S^3 and $\cdot$ for the inner product in $\mathbb{R}^3$, but we also omit the latter when the context is clear.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E_\sigma\left(\frac{u+t\varphi}{|u+t\varphi|}\right) &= \int_{S^3}\left\langle \nabla u, \nabla \frac{d}{dt}\Big|_{t=0}\left(\frac{u+t\varphi}{|u+t\varphi|}\right)\right\rangle + \sigma^2\Delta u\cdot\Delta\frac{d}{dt}\Big|_{t=0}\left(\frac{u+t\varphi}{|u+t\varphi|}\right) dvol_g \\
&= \int_{S^3}\langle\nabla u, \nabla(\varphi-(\varphi\cdot u)u)\rangle + \sigma^2\Delta u\cdot\Delta(\varphi-(\varphi\cdot u)u) dvol_g \\
&= \int_{S^3}\langle\nabla u, \nabla(P_u(\varphi))\rangle + \sigma^2\Delta u\cdot\Delta(P_u(\varphi)) dvol_g
\end{aligned}$$

Thus u solves the following equation in the distributional sense,

$$P_u(\Delta(u-\sigma^2\Delta u)) = 0 \quad\text{or}\quad \Delta(u-\sigma^2\Delta u) = \lambda u \tag{3.1}$$

with a priori only $\lambda \in W^{-2,2}(S^3,\mathbb{R})$. One may check that the above equation is also valid for harmonic maps (case $\sigma = 0$) into general target rather than a sphere or refer to the book of Moser [20] using the nearest point projection instead of P_u.

Now we calculate the formula for λ.

We know that for any $\phi \in C_c^\infty(S^3,\mathbb{R})$, $\varphi = \phi u \in W^{2,2}(S^3,\mathbb{R})$,

$$\int_{S^3}\lambda\phi\, dvol_g = \int_{S^3}\varphi\cdot\Delta(u-\sigma^2\Delta u)dvol_g = \int_{S^3}\phi(\Delta(u-\sigma^2\Delta u)\cdot u)dol_g$$

Thus integration by parts and using for the sphere $0 = \frac{1}{2}\nabla\cdot(\nabla(|u|^2)) = \Delta u\cdot u + |\nabla u|^2$, we have,

$$\begin{aligned}
\int_{S^3}\lambda\phi\, dvol_g &= \int_{S^3}\phi(\Delta(u-\sigma^2\Delta u)\cdot u)dol_g \\
&= \int_{S^3} -|\nabla u|^2\phi - \sigma^2\Delta u\cdot\Delta(u\phi)dvol_g \\
&= \int_{S^3} -|\nabla u|^2\phi - \sigma^2|\Delta u|^2\phi - \sigma^2\Delta u\cdot u\Delta\phi - 2\sigma^2\Delta u\cdot\langle\nabla u,\nabla\phi\rangle dvol_g \\
&= \int_{S^3} -|\nabla u|^2\phi - \sigma^2|\Delta u|^2\phi + \sigma^2\Delta|\nabla u|^2\phi + 2\sigma^2\nabla\cdot(\Delta u\cdot\nabla u)\phi\, dvol_g \\
&= \int_{S^3}\phi(-|\nabla u|^2 + \sigma^2|\Delta u|^2 + \sigma^2\Delta|\nabla u|^2 + 2\sigma^2\langle\nabla\Delta u,\nabla u\rangle)\, dvol_g
\end{aligned}$$

Thus the critical points satisfy the following equation in the distributional sense,

$$\Delta(u-\sigma^2\Delta u) = -|\nabla u|^2 u + \sigma^2 u(|\Delta u|^2 + \Delta|\nabla u|^2 + 2\langle\nabla\Delta u,\nabla u\rangle) \tag{3.2}$$

4. Palais-Smale Condition and Regularity of Critical Points

To apply the method of MinMax as structured by Palais [22], we still need to check that the functional $E_\sigma(u)$ satisfies the Palais-Smale condition.

Theorem 4.1. *For any $\sigma > 0$, the functional $E_\sigma(u)$ for $u \in W^{2,2}(S^3, S^2)$ satisfy the Palais-Smale condition, that is, given $(u_n)_{n\in\mathbb{N}} \in W^{2,2}(S^3, S^2)$ such that,*

$$DE_\sigma(u_n) \to 0, \quad E_\sigma(u_n) \to \beta_\sigma \in \mathbb{R} \tag{4.1}$$

there is a subsequence u_n (without relabeling) that converges to u in the $W^{2,2}(S^3, S^2)$ norm, which is equivalent to the convergence in the Finsler metric. It follows directly from the continuity of $DE_\sigma(\cdot)$ that $DE_\sigma(u) = 0$

Remark 4.2. In section 7, we would give a different proof of $DE_\sigma(u) = 0$ without using the continuity, but using the sphere structure of the target manifold.

Before proving this, we first prove an inequality that will be used often here. It has an analogy in Euclidean spaces (see Chapter 9 of Gilbarg-Trudinger [10] or the lecture notes of Giaquinta-Martinazzi [9]).

Lemma 4.3. *Assume $u \in W^{2,p}(M^m, \mathbb{R}^k), 1 < p < \infty$, where M^m is a closed smooth Riemannian manifold, then*

$$\int_M |\nabla^2 u|^p dvol_g \leq C \int_M (|\Delta u|^p + |\nabla u|^p) dvol_g$$

Proof. We may assume $k = 1$. Since M is compact, we can cover it with finitely many geodesic balls corresponds to a normal neighborhood, we write such a ball as $B_1 \subset B_2 \subset M$, with the B_1's cover M, and write $C_0 = \max(||g||_{C^1}, ||\Gamma||_\infty)$ to be the finite maximal on M of the coefficients of the metric and the Christoffel symbol in these coordinates. Then we have

$$\begin{aligned}
(\nabla^2 u)_{mn} &= \partial_{mn} u - \Gamma^r_{mn} \partial_r u \\
\int_{B_1} |\nabla^2 u|^p &= \int_{B_1} |g^{i_1 j_1} g^{i_2 j_2} (\nabla^2 u)_{i_1 i_2} (\nabla^2 u)_{j_1 j_2}|^{\frac{p}{2}} \\
&\leq C(C_0, m) \int_{B_1} \sum_{i,j} |\partial_{ij} u|^p + |\nabla u|^p
\end{aligned}$$

Now we only need to bound the first term of the right hand side of the last inequality,

$$\begin{aligned}
\int_{B_1} \sum_{i,j} |\partial_{ij} u|^p &\leq C \int_{B_2} \left| \sum_i \partial_{ii} u \right|^p \\
&\leq C(m, C_0) \int_{B_2} |\Delta u|^p + |\nabla u|^p
\end{aligned}$$

where the first inequality holds because of the Euclidean case using Calderón-Zygmund estimates [10], and the second inequality holds because we are in normal neighborhoods. □

Now we are ready to prove that the Palais-Smale condition holds.

Proof. Proof of Theorem 4.1.

Assume equation (4.1) holds, then since the functional $E_\sigma(u_n)$ converges and σ is fixed, we know that the sequence u_n has bounded $W^{2,2}$ norm independent of $n \in \mathbb{N}$, using the above lemma. Now by Eberlein-Šmulian we know that $u_n \in W^{2,2}(S^3, \mathbb{R}^3)$, a Hilbert space, has a weakly convergence subsequence. Again we know that the embedding $W^{2,2}(S^3, \mathbb{R}^3) \hookrightarrow W^{1,p}(S^3, \mathbb{R}^3)$ is compact for $p < 6$ and in particular also in $W^{1,2}(S^3, \mathbb{R}^3)$, which gives strong convergence in $W^{1,2}$ (up to choosing a subsequence again). Notice that the limit u lies in $W^{2,2}(S^3, S^2)$ due to strong C^0 convergence using Sobolev embedding again.

Now we need to show that we actually have strong convergence in $W^{2,2}(S^3, S^2)$. For this we need to use that the first variation of the energy converges to zero, which gives

$$\lim_{n\to\infty} \sup_{\|v\|_{W^{2,2}(S^3,\mathbb{R}^3)}\le 1} |DE_\sigma(u_n)P_{u_n}(v(x))| \to 0$$

$$\lim_{m,n\to\infty} \sup_{\|v\|_{W^{2,2}(S^3,\mathbb{R}^3)}\le 1} |DE_\sigma(u_n)P_{u_n}(v(x)) - DE_\sigma(u_m)P_{u_m}(v(x))| \to 0$$

By taking $v = v_{m,n} = u_n - u_m$ with $\|v\|_{W^{2,2}(S^3,\mathbb{R}^3)} \le 1$, for large $n, m \in \mathbb{N}$, our goal is to show,

$$\sigma^2 \int_{S^3} \Delta(u_n - u_m) \cdot \Delta v dvol_g = \sigma^2 \int_{S^3} \Delta(u_n - u_m) \cdot \Delta(u_n - u_m) dvol_g \to 0 \tag{4.2}$$

This corresponds to the last term in the following equation,

$$\begin{aligned}
&DE_\sigma(u_n)P_{u_n}(v) - DE_\sigma(u_m)P_{u_m}(v)\\
&= \int_{S^3} \langle \nabla(u_m - u_n), \nabla v\rangle - \langle \nabla u_m, \nabla\{(v\cdot u_m)u_m\}\rangle + \langle \nabla u_n, \nabla\{(v\cdot u_n)u_n\}\rangle dvol_g\\
&\quad - \sigma^2 \int_{S^3} \Delta(u_m)\cdot\Delta\{(v\cdot u_m)u_m\} dvol_g + \sigma^2 \int_{S^3} \Delta(u_n)\cdot\Delta\{(v\cdot u_n)u_n\} dvol_g \qquad (4.3)\\
&\quad + \sigma^2 \int_{S^3} \Delta(u_n - u_m)\cdot \Delta v \; dvol_g
\end{aligned}$$

Notice that by strong $W^{1,p}(p < 6)$ and L^∞ convergence, we have the terms without σ^2 converges to zero uniformly for $\|v\|_{W^{2,2}(S^3,\mathbb{R}^3)} \le 1$. Now we only need to deal with the terms in the middle line. Writing out we have

$$
\begin{aligned}
&\sigma^2 \int_{S^3} \langle \Delta(u_m), \Delta[(v\cdot u_m)u_m]\rangle dvol_g - \sigma^2 \int_{S^3} \langle \Delta u_n, \Delta[(v\cdot u_n)u_n]\rangle dvol_g \\
&= \sigma^2 \int_{S^3} \Delta u_m \cdot u_m \Delta(v\cdot u_m) - \Delta u_n \cdot u_n \Delta(v\cdot u_n) \qquad (4.4)\\
&\quad + \sigma^2 \int_{S^3} v\cdot u_n |\Delta u_n|^2 - v\cdot u_m |\Delta u_m|^2 \qquad (4.5)\\
&\quad + 2\sigma^2 \int_{S^3} \Delta u_n \cdot \langle \nabla u_n, \nabla(v\cdot u_n)\rangle - \Delta u_m \cdot \langle \nabla u_m, \nabla(v\cdot u_m)\rangle
\end{aligned}
$$

We bound the last term of the above equation using integration by parts,

$$
\begin{aligned}
&2\sigma^2 \Big(\int_{S^3} \Delta u_n \cdot \langle \nabla u_n, \nabla(v\cdot u_n)\rangle - \Delta u_m \cdot \langle \nabla u_m, \nabla(v\cdot u_m)\rangle\Big) \\
&= 2\sigma^2 \int_{S^3} \Delta(u_n - u_m)\cdot \langle \nabla u_n, \nabla(v\cdot u_n)\rangle + \Delta u_m \cdot \langle \nabla(u_n-u_m), \nabla(v\cdot u_n)\rangle \\
&\quad + 2\sigma^2 \int_{S^3} \Delta u_m \cdot \langle \nabla u_m, \nabla(v\cdot(u_n-u_m))\rangle \\
&\leq 2\sigma^2 \int_{S^3} -\langle \nabla(u_n-u_m), \nabla\langle \nabla u_n, \nabla(v\cdot u_n)\rangle\rangle \\
&\quad + 2\sigma^2 \|\Delta u_m\|_{L^2} \|\nabla(v\cdot u_n)\|_{L^4} \|\nabla(u_n-u_m)\|_{L^4} \\
&\quad + 2\sigma^2 \|\Delta u_m\|_{L^2} \|\nabla u_m\|_{L^4} (\|u_n-u_m\|_{L^\infty} \|\nabla v\|_{L^4} + \|\nabla(u_n-u_m)\|_{L^4} \|v\|_{L^\infty}) \\
&\to 0 \quad (\text{as } m,n\to\infty)
\end{aligned}
$$

The term in equation (4.5) have the problem of non-linearity, we bound it by the choice of $v = u_n - u_m$ with the embedding $W^{2,2}(S^3, \mathbb{R}^3) \hookrightarrow L^\infty$.

$$
\begin{aligned}
&\sigma^2 \int_{S^3} v\cdot u_n |\Delta u_n|^2 - v\cdot u_m |\Delta u_m|^2 \\
&= \sigma^2 \int_{S^3} v\cdot (u_n-u_m)|\Delta u_n|^2 + v\cdot u_m(|\Delta u_n|^2 - |\Delta u_m|^2) \\
&= \sigma^2 \int_{S^3} (u_n-u_m)\cdot(u_n-u_m)|\Delta u_n|^2 + (u_n-u_m)\cdot u_m(|\Delta u_n|^2 - |\Delta u_m|^2) \\
&\leq C\sigma^2 (\|\Delta u_n\|_{L^2} + \|\Delta u_m\|_{L^2}) \|u_n-u_m\|_{L^\infty} \to 0 \quad (\text{as } m,n\to\infty)
\end{aligned}
$$

One can use the equality $\Delta u_m \cdot u_m + |\nabla u_m|^2 = 0$ and integration by parts to obtain convergence of the term in equation 4.4,

$$
\begin{aligned}
&\sigma^2 \int_{S^3} \Delta u_m \cdot u_m \Delta(v \cdot u_m) - \Delta u_n \cdot u_n \Delta(v \cdot u_n) \\
=&\sigma^2 \int_{S^3} -|\nabla u_m|^2 \Delta(v \cdot u_m) + |\nabla u_n|^2 \Delta(v \cdot u_n) \\
=&\sigma^2 \int_{S^3} (|\nabla u_n|^2 - |\nabla u_m|^2)\Delta(v \cdot u_n) + |\nabla u_m|^2 \Delta(v \cdot (u_n - u_m)) \\
=&\sigma^2 \int_{S^3} (|\nabla u_n|^2 - |\nabla u_m|^2)\Delta(v \cdot u_n) - 2\sigma^2 \int_{S^3} \langle\langle \nabla^2 u_m, \nabla u_m\rangle, \nabla(v \cdot (u_n - u_m))\rangle \\
\leq& C\sigma^2 \|\nabla(u_n - u_m)\|_{L^4} (\|\nabla u_n\|_{L^4} + \|\nabla u_m\|_{L^4})(\|\Delta v\|_{L^2} + \|\Delta u_n\|_{L^2}\|v\|_{L^\infty} + \|\nabla v\|_{L^4}^2 + \|\nabla u_n\|_{L^4}^2) \\
&+ C\sigma^2 (\|\nabla^2 u_m\|_{L^2} \|\nabla u_m\|_{L^4})(\|\nabla v\|_{L^4} \|u_n - u_m\|_{L^\infty} + \|\nabla(u_n - u_m)\|_{L^4}\|v\|_{L^\infty}) \\
&\to 0 \quad (\text{as } m, n \to \infty)
\end{aligned}
$$

Thus we have shown that equation (4.3) converges to zero and thus completed equation (4.2): u_n is Cauchy in $W^{2,2}$ and the limit of strong convergence and weak convergence in $W^{2,2}$ should coincide, i.e. u_n converges strongly to a critical point u in $W^{2,2}(S^3, S^2)$. □

Theorem 4.4. *Critical points of E_σ, satisfying equation (3.1), is smooth.*

Proof. We split the PDE into divergence and non-divergence terms,

$$
\begin{aligned}
\Delta(-u + \sigma^2 \Delta u) =& |\nabla u|^2 u - \sigma^2 u(|\Delta u|^2 + \Delta|\nabla u|^2 + 2\langle \nabla \Delta u, \nabla u\rangle) \\
=& |\nabla u|^2 u - \sigma^2 |\Delta u|^2 u \\
&- 2\sigma^2 \nabla \cdot (\langle \nabla^2 u, \nabla u\rangle u) + 2\sigma^2 \langle\langle \nabla^2 u, \nabla u\rangle, \nabla u\rangle \\
&- 2\sigma^2 \nabla \cdot [(\Delta u \cdot \nabla u)u] + 2\sigma^2 \langle \Delta u \cdot \nabla u, \nabla u\rangle + 2\sigma^2 |\Delta u|^2 u \\
=& (|\nabla u|^2 + \sigma^2 |\Delta u|^2)u + 2\sigma^2(\langle \Delta u \cdot \nabla u, \nabla u\rangle + \langle\langle \nabla^2 u, \nabla u\rangle, \nabla u\rangle) \\
&- 2\sigma^2 \nabla \cdot (\langle \nabla^2 u, \nabla u\rangle u + [(\Delta u \cdot \nabla u)u]) \\
=& A + \nabla \cdot (B)
\end{aligned}
$$

Here we collected all the divergence terms into $B \in L^{\frac{3}{2}}$ and the rest into $A \in L^1$.

Since the domain S^3 is smooth, to treat regularity for this PDE we may work in normal coordinates and view the operators as a perturbation of the flat Laplacian (by rotation, scaling and freezing the coefficients, one may refer to Gilbarg-Trudinger [10] for details in case $p \in (1, \infty)$). For simplicity, we show for the flat case.

We apply a partition of unity argument so that we consider the local estimates of the two problems,

$$(\mathrm{Id} - \sigma^2 \Delta u_1)\Delta(u_1) = A \text{ on } B_2(0)$$
$$u_1 = 0 \text{ on } \partial B_2(0)$$

$$(\mathrm{Id} - \sigma^2 \Delta u_2)\Delta(u_2) = \nabla \cdot (B) \text{ on } B_2(0)$$
$$u_2 = 0 \text{ on } \partial B_2(0)$$

We first estimate the second problem using the lemma ([10]), for $v \in W^{2,2}(S^3, \mathbb{R})$,

$$\Delta^2 v = div(F) \implies \|\nabla^3 v\|_{L^q} \leq C\|F\|_{L^q}, 1 < q < \infty \tag{4.6}$$

For $q = \frac{3}{2}$, this gives us $\nabla^3 u_2 \in L^{\frac{3}{2}}$ and thus $B \in L^2$– one can iterate this process and get $\nabla^3 u_2 \in L^q$, $q < \infty$. Notice in the non-flat case we can use the equation again to conclude $\nabla^2 \Delta u_2 \in L^q$, together with $\|\nabla^2 \Delta u_2 - \Delta \nabla^2 u_2\|_{L^q} \leq C\|\nabla^3 u_2\|_{L^q}$ we get $\nabla^4 u_2 \in L^q$. Now we can differentiate with respect to the second equation, and use $\|\nabla\{(\mathrm{Id} - \sigma^2 \Delta u_2)\Delta(u_2)\} - (\mathrm{Id} - \sigma^2 \Delta u_2)\Delta(\nabla u_2)\|_{L^q} \leq C\|\nabla^4 u_2\|_{L^q}$ to iterate with respect to higher order derivatives, and finally conclude that u_2 is smooth.

We look at the first equation, $A \in L^1$ implies $\nabla^2 \Delta u_1 \in L^{1,\infty}$(see Corollary 6.1.6 [11]), which gives us $\Delta u_1 \in L^p \cap L^2$(p<3)– for this one may check the interpolation for Lorentz spaces in [18], we can again interpolate and iterate to get $\nabla^2 u_1 \in L^q$, $q < \infty$, also $\nabla^2 \Delta u_1 \in L^q$. Now $\|\nabla^3 u_1\|_{L^q} \leq C(\|\Delta \nabla u_1\|_{L^q} + \|\nabla^2 u_1\|_{L^q}) \leq C(\|\nabla \Delta u_1\|_{L^q} + \|\nabla^2 u_1\|_{L^q})$, and $\|\nabla^4 u_1 - \nabla^2 \Delta u_1\|_{L^q} \leq C\|\nabla^3 u_1\|_{L^q}$. Differentiate the equation again and apply a similar argument as for the second equation, we get finally u_1 is smooth. □

5. Admissible Family

The definition of an admissible family depends on the flow one uses on the Banach manifold. For more examples one can refer to the lecture notes by Palais [21] or the book by Struwe [30].

We set the notation $\overline{f(\cdot)} := \fint_D f(x)dx$ to be the mean integral of the function over its domain. Consider the following family, where B^4 is the open unit ball in $\mathbb{R}^4$,

$$\mathcal{A} = \Big\{ u \in C^0(\overline{B^4}, W^{2,2}(S^3, S^2)) \Big| \max_{\bar{x}\in\partial B^4} \int_{S^3} |\nabla u(\bar{x},\cdot)|^2 + \sigma_0^2 |\Delta u(\bar{x},\cdot)|^2 \leq \frac{1}{2} C_{S^3}, \\ \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|} \text{ is not null-homotopic from } S^3 \text{ to } S^2 \Big\}$$

The σ_0 will be chosen later to be small so that our admissible family is not empty. Notice that we make the family to be independent of σ but the energy will still make the flow dependent on σ.

The constant C_{S^3} is determined in the following lemma.

Lemma 5.1. *There is a constant C_{S^3}>0, such that for any $v \in W^{1,2}(S^3, S^2)$, if $\int_{S^3} |\nabla v|^2 \leq C_{S^3}$, then $\bar{v} = |\fint_{S^3} v| \in B^4_1(0) \setminus B^4_{1/2}(0)$.*

Proof. Indeed, applying Poincaré inequality we have,

$$\fint_{S^3} |v - \bar{v}|^2 dvol_g \leq C_0 \int_{S^3} |\nabla v|^2 dvol_g \leq C_0 C_{S^3}$$

thus there is some point $y \in S^3$ such that $|v(y) - \bar{v}| \leq C_0 C_{S^3}$ and since $v(y) \in S^2$, choosing C_{S^3} such that $C_0 C_{S^3} \leq 1/4$, we get $\bar{v} \in B^4_1(0) \setminus B^4_{1/2}(0)$ as claimed. □

Now let us define the width with respect to σ of our problem.

Definition 5.2. *We call the following width with respect to σ,*

$$\beta(\sigma) := \inf_{u\in\mathcal{A}} \max_{x\in\overline{B^4}} E_\sigma(u(x,\cdot)) = \inf_{u\in\mathcal{A}} \max_{x\in\overline{B^4}} \int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2 dvol_g$$

Back to our family $\mathcal{A}$, we first assume it's not empty, and show that its width is strictly positive.

Lemma 5.3. $\beta(\sigma) \geq C_{S^3} > 0$.

Proof. Towards a contradiction, we assume there is $u \in \mathcal{A}$ such that $\max_{x\in\overline{B^4}} E_\sigma(u) < C_{S^3}$. Then we know that $\overline{u(x,\cdot)} \in B^4_1(0) \setminus B^4_{1/2}(0)$, and is a continuous function with respect to x. Indeed, for any $\epsilon > 0$, if $|x_1 - x_2| \leq \delta(\epsilon)$, with $\delta(\epsilon)$ given by continuity of $u(\cdot, z)$,

$$|u(x_1,\cdot) - u(x_2,\cdot)|_{L^\infty} \leq |u(x_1,\cdot) - u(x_2,\cdot)|_{W^{2,2}} \leq \epsilon$$

Thus $\fint_{S^3} |u(x_1,\cdot) - u(x_2,\cdot)| \leq |u(x_1,\cdot) - u(x_2,\cdot)|_{L^\infty} \leq \epsilon$, and we have continuity as claimed. It follows that $\frac{\overline{u(x,\cdot)}}{|\overline{u(x,\cdot)}|}$ is also continuous with respect to x. Similarly, $\frac{\overline{\phi(\bar{x},\cdot)}}{|\overline{\phi(\bar{x},\cdot)}|}$ is a continuous not null-homotopic function, which is a contradiction to the strong $W^{2,2}$ convergence of $u(x,\cdot)$ to $\phi(\bar{x},\cdot)$, since $|\fint_{S^3} u(\lambda\bar{x},\cdot)-\phi(\bar{x},\cdot)| \le C\fint_{S^3}|u(\lambda\bar{x},\cdot)-\phi(\bar{x},\cdot)| \le C(\fint_{S^3}|u(\lambda\bar{x},\cdot)-\phi(\bar{x},\cdot)|^2)^{\frac{1}{2}} \to 0$ (the only type of continuous extension of functions on S^3 onto $\bar{B}^4$ are the ones homotopic to a constant function). □

To build an example of a function in our family for small σ (to be decided later), we first consider the following conformal map. Let $a \in B^4$,

$$\varphi_a(x) = (1-|a|^2)\frac{x-a}{|x-a|^2} - a$$

One may check that this is inversion through a sphere with radius $\sqrt{1-|a|^2}$ and center a combined with a translation. Indeed, let us define $y = \varphi_a(x)$ and $w = y + 2a$, then we have,

$$\frac{w-a}{1-|a|^2} = \frac{x-a}{|x-a|^2}$$

And the above is the generalization of the two dimensional formula of inversion through a sphere to higher dimensions. For a complete derivation, please refer to the book of Blair [2].

One can check that the map φ_a sends S^3 to S^3, and is conformal by the isometric embedding of $S^3 \hookrightarrow \mathbb{R}^4$.

Lemma 5.4. *For $p < 3$, let $C_{\bar{x}}$ be the constant map that send S^3 to $-\bar{x}$ on S^3, then as $\lambda \to 1$, $\|\varphi_{\lambda\bar{x}} - C_{\bar{x}}\|_{W^{1,p}} \to 0$.*

Proof. We only prove here $\|\nabla(\varphi_{\lambda\bar{x}} - C_{\bar{x}})\|_{L^p} \to 0$. This is where the condition $p < 3$ is needed, one may check through a similar argument that $\|\varphi_{\lambda\bar{x}} - C_{\bar{x}}\|_{L^p} \to 0$ independent of this condition (actually for any $p < \infty$). Choosing $a = \lambda\bar{x}$,

$$\partial_i(\varphi_a^k)(x) = \frac{1-\lambda^2}{|x-\lambda\bar{x}|^4}(\delta_{ik}|x-\lambda\bar{x}|^2 - 2(x_i-\lambda\bar{x}_i)(x_k-\lambda\bar{x}_k))$$

$$|\nabla\varphi_a|^2 = \sum_{i,k}|\partial_i(\varphi_a^k)(x)|^2 = \frac{(1-\lambda^2)^2}{|x-\lambda\bar{x}|^4}$$

Thus we may integrate and we only need to check the integration in the geodesic ball $B_\delta(\bar{x})$ on S^3 for some small δ.

$$\begin{aligned}\int_{B_\delta(\bar{x})} |\nabla\varphi_a|^p dvol_g =& |1-\lambda^2|^p \int_{B_\delta(\bar{x})} \frac{1}{|x-\lambda\bar{x}|^{2p}} \\ \leq& C|1+\lambda|^p|1-\lambda|^p \int_0^\delta \frac{r^2\,dr}{(r^2+|1-\lambda|^2)^p} \\ \leq& C|1-\lambda|^p \int_0^{\delta'} \frac{|1-\lambda|^3}{|1-\lambda|^{2p}} \frac{r^2\,dr}{|r^2+1|^p} \\ \leq& C|1-\lambda|^{3-p}\end{aligned}$$

here $\delta' = \frac{\delta}{|1-\lambda|}$ after a change of variable. As $\lambda \to 1$ the integral converges and with the above inequality, the claim follows.

Notice that we used the property that in the geodesic ball $B_\delta(\bar{x})$ for δ small enough, the geodesic distance on S^3 between two points are comparable to the Euclidean distance. Using this one can actually show a reverse of the above inequality and obtain that the convergence is no longer true for $p \geq 3$.

Also notice that we calculated the derivatives of the map as from the flat Euclidean space to Euclidean space, but because of the isometric embedding of S^3 into the Euclidean space, this is enough to bound the $W^{1,p}$ convergence. □

We can compose the map φ_a with any smooth, not null-homotopic map h from S^3 to S^2, for example the Hopf fibration (for an elementary introduction see [17] or a more complete reference in [3]). Now assume $\bar{u}(x,z) = h \circ \varphi_x(z)$, for fixed $x \in B^4$, the map $\varphi_x(z)$ is smooth on S^3, and the derivatives of $\varphi_x(z)$ is continuous with respect to $x \in B^4$. However the smoothness of the map does not go up to the boundary ∂B^4. Also notice that in the proof of the above lemma, the convergence is independent of $\bar{x}$, so in fact we have

$$\lim_{\lambda\to 1} \max_{\bar{x}\in\partial B^4} \|\varphi_{\lambda\bar{x}}(x) - C_{\bar{x}}\|_{W^{1,2}} \to 0$$

This allows us to choose λ close to 1 so that,

$$\max_{\bar{x}\in\partial B^4} \|\nabla(\varphi_{\lambda\bar{x}}(x) - C_{\bar{x}})\|_{L^2} \leq \min\left\{\left(\frac{1}{4}C_{S^3}\right)^{\frac{1}{2}}, \frac{1}{2C_0}\right\}$$

for C_0 to be decided below.

Furthermore, we can choose $\sigma \leq \sigma_0$ so that,

$$\max_{\bar{x}\in\partial B^4} \sigma^2 \int_{S^3} |\Delta\bar{u}(\lambda\bar{x},\cdot)|^2 \leq \frac{1}{4}C_{S^3}$$

Altogether, now we can define $u(x,z) = \bar{u}(\lambda x, z)$, and we get from the above inequalities that,

$$\max_{\bar{x}\in\partial B^4} \int_{S^3} |\nabla u(\bar{x},\cdot)|^2 + \sigma^2 |\Delta u(\bar{x},\cdot)|^2 \leq \frac{1}{2} C_{S^3}$$

It remains to show that $\frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|}$ is not null-homotopic. Notice,

$$\begin{aligned}\max_{\bar{x}\in\partial B^4} |\fint_{S^3} u(\bar{x},\cdot) - h(-\bar{x})| \leq & C \max_{\bar{x}\in\partial B^4} \int_{S^3} |u(\bar{x},\cdot) - h(-\bar{x})| \\ \leq & C \max_{\bar{x}\in\partial B^4} \int_{S^3} |\varphi_{\lambda\bar{x}} - C_{\bar{x}}| \\ \leq & C_0 \max_{\bar{x}\in\partial B^4} \|\varphi_{\lambda\bar{x}}(x) - C_{\bar{x}}\|_{W^{1,2}}.\end{aligned}$$

Here the constant C_0 depends on the first derivatives of the map h (notice the distances we used here are all Euclidean as in $S^2 \hookrightarrow \mathbb{R}^3$ and $S^3 \hookrightarrow \mathbb{R}^4$, because the Hopf map can be defined smoothly on the whole $\mathbb{R}^4$). Thus, using the geometric fact that if a point p lies in $B_{\frac{1}{2}}(q)$ with $q \in S^2$, then $|\frac{p}{|p|} - q| \leq |p - q|$, we have

$$\max_{\bar{x}\in\partial B^4} \left| \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|} - h(-\bar{x}) \right| \leq \max_{\bar{x}\in\partial B^4} \left| \overline{u(\bar{x},\cdot)} - h(-\bar{x}) \right| \leq \frac{1}{2}$$

Hence we can build a homotopy directly from $f_1 = \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|}$ to $f_2 = h(-\bar{x})$ by $f = \frac{tf_1 + (1-t)f_2}{|tf_1 + (1-t)f_2|}$. By assumption the map f_2 is not null-homotopic, and this shows that $f_2 = \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|}$ is also not null-homotopic.

Finally, we need to show the family is well-defined for the Min-Max procedure.

First of all, the family is invariant under the (forward) pseudo gradient vector flow $\Phi(t,u)$, with $X(u)$ a pseudo gradient vector field on the Finsler manifold $W^{2,2}(S^3,S^2)$ constructed as in Palais' paper [22] (a more detailed treatment about the structure and estimates relating the pseudo gradient flow is also given in the proofs of section 6),

$$\partial_t \Phi(t,u) = -X(u)\eta(u), \quad \Phi(0,u) = u, \quad u \in W^{2,2}(S^3,S^2)$$

Here $0 \leq \eta(u) \leq 1$ is supported on

$$\left\{ u \in W^{2,2}(S^3,S^2), E_\sigma(u) = \int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2 \geq \frac{3}{4} C_{S^3} \right\}$$

Then $\Phi(t,u)(t > 0)$ is a diffeomorphism on $W^{2,2}(S^3,S^2)$ and the energy is non-increasing with respect to t.

For any $u(x,z) \in \mathcal{A}$, $(\Phi \circ u)(t,x,z)$ is continuous with respect to $x \in \overline{B^4}$, and

$$\max_{\bar{x}\in\partial B^4} E_\sigma(\Phi \circ u(\bar{x},\cdot)) \leq E_\sigma(u(\bar{x},\cdot)) \leq \frac{1}{2} C_{S^3}$$

Also $\frac{\overline{\Phi\circ u(\bar{x},\cdot)}}{|\overline{\Phi\circ u(\bar{x},\cdot)}|} = \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|}$ is not null-homotopic since the flow leaves $u(\bar{x},\cdot)$ invariant.

In total, we have built an admissible family $\mathcal{A}$ for the energy $E_\sigma(u) = \int_{S^3} |\nabla u|^2 + \sigma^2|\Delta u|^2$ on the Finsler manifold $W^{2,2}(S^3, S^2)$. We know that it is not empty, and from Lemma 5.3 that the width $\beta(\sigma)$ is strictly positive. Also by the example we give and by construction our width is finite and hence we can apply the MinMax method as in Palais' paper, and obtain that there is a critical point for each $\sigma \leq \sigma_0$ at the value $\beta(\sigma)$,

$$E_\sigma(u) = \inf_{u\in\mathcal{A}} \max_{x\in\overline{B^4}} E_\sigma(u(x,\cdot)) = \beta(\sigma), \quad DE_\sigma(u) = 0$$

6. Entropy Condition

Notice that at the critical points constructed above, we can show that $\beta_\sigma \to \beta_0$ for all $\sigma \to 0$. Indeed, for any $\epsilon > 0$, we can pick $u \in \mathcal{A}$ such that,

$$\sup_{x\in\overline{B^4}} E_0(u(x,\cdot)) \leq \beta(0) + \epsilon$$

Since $\overline{B^4}$ is compact, we know that the $W^{2,2}$-norm of u is uniformly bounded with respect to $x \in \overline{B^4}$. Hence we can choose σ small enough so that,

$$\sup_{x\in\overline{B^4}} E_\sigma(u(x,\cdot)) \leq \sup_{x\in\overline{B^4}} E_0(u(x,\cdot)) + \epsilon \leq \beta(0) + 2\epsilon$$

Let $\epsilon \to 0$ we then get the claim.

However we don't know if the viscosity term behaves well, that is, do we have the following convergence for the critical points of the Min-Max procedure,

$$\lim_{\sigma\to 0} \sigma^2 \int_{S^3} |\Delta u|^2 \to 0 \quad ?$$

What we can show is that we can obtain a sequence of $\sigma_n \to 0$ such that the above is true for some critical points of E_{σ_n}. This can be seen firstly in the paper of Struwe [29], also in the lecture notes of Rivière [26]. The proof has a few elements and we first state them altogether.

Step 1. We first estimate the derivatives of $\beta(\sigma)$ and E_σ to obtain, at almost every $\sigma \geq 0$,

$$\liminf_{\sigma\to 0} \sigma \log(\frac{1}{\sigma})\beta'(\sigma) = 0 \tag{6.1}$$

Step 2. First we fix $\epsilon > 0$ and $\sigma > 0$, we show that for any sequence $\sigma_k \to \sigma^+$, we can find a $u = u_k(x,z) \in W^{2,2}(S^3, S^2)$ (the choice depends on ϵ and σ_k) such that

$$E_\sigma(u) \geq \beta(\sigma) - \epsilon(\sigma_k - \sigma) \tag{6.2}$$

$$E_{\sigma_k}(u) \leq \beta(\sigma_k) + \epsilon(\sigma_k - \sigma) \tag{6.3}$$

$$\partial_\sigma E_\sigma(u) \leq \beta'(\sigma) + 3\epsilon \tag{6.4}$$

Step 3. Assume we have such a family: $\sup_{u\in\mathcal{F}} \int_{S^3} |\nabla u|^2 + \sigma^2 |\Delta u|^2 \leq C_0 < \infty$, then

$$\sup_{u\in\mathcal{F}} \|DE_{\sigma_k}(u) - DE_\sigma(u)\| \leq C_0 C(\sigma_k - \sigma) \tag{6.5}$$

Using this condition, if we can find a $u \in W^{2,2}(S^3, S^2)$ satisfying the above equations (6.2) to (6.4) , then there is a sequence u_k such that,

$$\|DE_{\sigma_k}(u_k)\| \to 0$$

Step 4. We collect Step 1 to Step 3 altogether and select ϵ and σ properly to obtain a sequence $\sigma_n \to 0$, and $u_n \in W^{2,2}(S^3, S^2)$ such that

$$((\partial_\sigma E_\sigma))_{\sigma=\sigma_n}(u_n) = \frac{o(1)}{\sigma_n \log(\frac{1}{\sigma_n})} \quad \text{and thus} \quad \lim_{\sigma_n\to 0} \sigma_n^2 \int_{S^3} |\Delta u_n|^2 \to 0 \tag{6.6}$$

Proof. Let's start proving them step by step. Recall by definition,

$$\beta_\sigma = E_\sigma(u) = \inf_{u\in\mathcal{A}} \sup_{x\in\overline{B^4}} E_\sigma(u)$$

Step 1 is straightforward. As $\beta(\sigma)$ is monotone, Lebesgue's theorem implies differentiability almost everywhere. We argue by contradiction, assume that $\sigma \log(\frac{1}{\sigma})\beta'(\sigma)$ is bounded from below by $\delta > 0$ as $\sigma \to 0$,

$$\beta(\sigma) - \beta(0) = \int_0^\sigma \beta'(x)dx \geq \int_0^\sigma \frac{\delta}{\sigma \log(\frac{1}{\sigma})} = \infty$$

Step 2 is contained in the following lemma.

Lemma 6.1. *Assume β is differentiable at σ, for fixed $\epsilon > 0$, $\sigma_k \to \sigma^+$, there is a $u = u_k(x,z) \in W^{2,2}(S^3, S^2)$,*

$$\beta(\sigma) - \epsilon(\sigma_k - \sigma) \leq E_\sigma(u) \leq E_{\sigma_k}(u) \leq \beta(\sigma_k) + \epsilon(\sigma_k - \sigma)$$

$$\partial_\sigma E_\sigma(u) \leq \beta'(\sigma) + 3\epsilon$$

Proof. By definition of $\beta(\sigma)$, we have that there is a $u(x,z) \in \mathcal{A}$,

$$\sup_{x\in B^4} E_{\sigma_k}(u) \le \beta(\sigma_k) + \epsilon(\sigma_k - \sigma) \tag{6.7}$$

Now for such $u(x,z) \in \mathcal{A}$, we may choose $x \in B^4$ so that $u(\cdot) = u(x,\cdot)$ has,

$$E_\sigma(u) \ge \beta(\sigma) - \frac{\epsilon}{2}(\sigma_k - \sigma)$$

Notice the choice of u depends both on ϵ and σ_k. Combining with the differentiability and monotonicity, for k large enough,

$$\beta(\sigma_k) \le \beta(\sigma) + (\beta'(\sigma) + \epsilon)(\sigma_k - \sigma)$$

we get the first claimed inequality,

$$\beta(\sigma) - \frac{\epsilon}{2}(\sigma_k - \sigma) \le E_\sigma(u) \le E_{\sigma_k}(u) \le \beta(\sigma) + (\beta'(\sigma) + 2\epsilon)(\sigma_k - \sigma)$$

$$\frac{E_{\sigma_k}(u) - E_\sigma(u)}{\sigma_k - \sigma} \le \beta'(\sigma) + \frac{5}{2}\epsilon$$

Since E_σ is C^1 with respect to σ, also $\partial_\sigma E_\sigma$ is monotone, as $k \to \infty$, we have

$$\partial_\sigma(E_\sigma(u_k)) \le \frac{E_{\sigma_k}(u_k) - E_\sigma(u_k)}{\sigma_k - \sigma} + \frac{\epsilon}{2}$$

this then gives us the second claimed inequality, writing $u = u_k$,

$$\partial_\sigma(E_\sigma(u)) \le \beta'(\sigma) + 3\epsilon$$

□

The first claim in Step 3 can be obtained from the estimates below, for fixed $\sigma, \epsilon > 0$ and $\sigma_k \to \sigma^+$,

$$\begin{aligned} \|DE_{\sigma_k}(u) - DE_\sigma(u)\| &= \sup_{\|v\|_{W^{2,2}\le 1}} (\sigma_k^2 - \sigma^2)\int_{S^3} \langle \Delta u, \Delta v\rangle dvol_g \\ &\le (\sigma_k + \sigma)(\sigma_k - \sigma)(\int_{S^3} |\Delta u|^2 dvol_g)^{\frac{1}{2}} \\ &\le (\sigma_k - \sigma)(1 + \frac{\sigma_k}{\sigma})(\sigma^2 \int_{S^3} |\Delta u|^2 dvol_g)^{\frac{1}{2}} \\ &\le C(\sigma_k - \sigma) \to 0 \end{aligned}$$

The second claim in Step 3 is contained in the following lemma.

Lemma 6.2. *We assume for σ and σ_k as of the previous lemma, then there is a sequence $u_k \in W^{2,2}(S^3, S^2)$ also satisfying equation (6.2) to (6.4), such that*

$$\|DE_{\sigma_k}(u_k)\| \to 0$$

Proof. We argue by contradiction, assume there is some $\delta > 0$ such that for all $u = u(x,z) \in W^{2,2}(S^3, S^2)$ satisfying equation (6.2) and (6.3),

$$\liminf_{k\to\infty} \|DE_{\sigma_k}(u)\| \geq \delta$$

Since $\mathcal{M} = W^{2,2}(S^3, S^2)$ is a Finsler manifold, we write

$$\mathcal{M}_k^* = \left\{u \in W^{2,2}(S^3, S^2), \|DE_{\sigma_k}(u)\| \neq 0\right\}$$

then for each E_{σ_k}, there is a pseudo gradient vector field X_k on $\mathcal{M}_k^*$ such that,

$$\|X_k\| \leq 2\|DE_{\sigma_k}\|, \quad DE_{\sigma_k}(X_k) \geq \|DE_{\sigma_k}\|^2$$

And we consider the sequence of flows on $\mathcal{M}_k^*$,

$$\begin{aligned}
\bar{X}_k(u) &:= \varphi\left(\frac{E_\sigma(u) - \beta(\sigma) + \epsilon(\sigma_k - \sigma)}{\epsilon(\sigma_k - \sigma)}\right) X_k(u) \\
\frac{d\phi_t^k(u)}{dt} &= -\bar{X}_k(\phi_t^k(u)) \\
\phi_0^k(u) &= u
\end{aligned}$$

Here $\varphi(\cdot)$ is a smooth increasing cut-off function supported on $\mathbb{R}_{\geq 0}$, strictly positive on $\mathbb{R}_{>0}$ and equal to 1 on $\mathbb{R}_{\geq 1}$.

Clearly E_{σ_k} is non-increasing with respect to the flow $\phi_t^k(u)$. Now we estimated for E_σ,

$$\begin{aligned}
\frac{dE_\sigma(\phi_t^k(u))}{dt} =& - DE_\sigma\left(\frac{E_\sigma(u) - \beta(\sigma) + \epsilon(\sigma_k - \sigma)}{\epsilon(\sigma_k - \sigma)}\right)(\bar{X}_k(\phi_t^k(u))) \\
=& - \varphi\left(\frac{E_\sigma(\phi_t^k(u)) - \beta(\sigma) + \epsilon(\sigma_k - \sigma)}{\epsilon(\sigma_k - \sigma)}\right) DE_\sigma(\phi_t^k(u))(X_k(\phi_t^k(u))) \\
=& - \varphi_k(\phi_t^k(u)) DE_{\sigma_k}(\phi_t^k(u))(X_k(\phi_t^k(u))) \\
&+ \varphi_k(\phi_t^k(u))[DE_{\sigma_k}(\phi_t^k(u)) - DE_\sigma(\phi_t^k(u))](X_k(\phi_t^k(u)))
\end{aligned}$$

We write

$$\varphi_k(\phi_t^k(u)) = \varphi\left(\frac{E_\sigma(\phi_t^k(u)) - \beta(\sigma) + \epsilon(\sigma_k - \sigma)}{\epsilon(\sigma_k - \sigma)}\right)$$

If we start at an initial point u satisfying (6.2), whose existence is guaranteed by the previous lemma, then

$$E_\sigma(\phi_t^k(u)) \leq E_{\sigma_k}(\phi_t^k(u)) \leq \beta(\sigma_k) + \epsilon(\sigma_k - \sigma)$$

then $E_\sigma(\phi_t^k(u))$ is bounded from above within the maximal time $T_k(u)$. Notice if

$$E_\sigma(\phi_t^k(u)) - \beta(\sigma) + \epsilon(\sigma_k - \sigma) < 0$$

then the flow is constant, and we get $E_\sigma(\phi_t^k(u))$ is bounded from below within the maximal time $T_k(u)$.

Now recalling equation (6.5) for this flow, we can choose k large enough so that,

$$||DE_{\sigma_k}(u) - DE_\sigma(u)|| \le C(\beta_\sigma, \epsilon)(\sigma_k - \sigma) \le \frac{\delta}{4}$$

Thus,

$$\frac{dE_\sigma(\phi_t^k(u))}{dt} \le -\varphi_k(\phi_t^k(u))\delta^2 + \varphi_k(\phi_t^k(u))||X_k(\phi_t^k(u))||\frac{\delta}{4} \le -\varphi_k(\phi_t^k(u))\frac{\delta^2}{2}$$

Hence E_σ is also decreasing along the flow.

In particular, since $E_\sigma(\phi_t^k(u))$ is continuous, if the flow reaches below the level $\beta(\sigma) - \epsilon(\sigma_k - \sigma)$, then it must also reach the level $\beta(\sigma) + \epsilon(\sigma_k - \sigma)$ from above. Also since we know that below this level,

$$\frac{dE_\sigma(\phi_t^k(u))}{dt} = 0$$

thus the flow becomes stationary. This will give,

$$E_\sigma(\phi_t^k(u)) = \beta(\sigma) - \epsilon(\sigma_k - \sigma) < \beta(\sigma)$$

Recall our notation again in Lemma 6.1– since $\phi_t^k(u) = \phi_t^k(u(x,z)) \in \mathcal{A}$ and we have required equation (6.7) for the initials, then

$$\sup_{x\in\bar{B}^4} E_\sigma(\phi_t^k(u)) < \beta(\sigma)$$

this is a contradiction to the definition of $\beta(\sigma)$.

Another useful knowledge is that the maximal time $T_k(u) = \infty$. Here we can apply the following estimates of the Finsler distance in Palais' paper [22],

$$\begin{aligned}
d(\phi_{t_1}^k(u), \phi_{t_2}^k(u)) &\le \int_{t_1}^{t_2} \left\|\frac{d}{dt}\phi_t^k(u)\right\| dt \\
&= \int_{t_1}^{t_2} \left\|\bar{X}_k(\phi_t^k(u))\right\| dt \\
&\le 2\int_{t_1}^{t_2} ||DE_{\sigma_k}(\phi_t^k(u))||dt \\
&\le 2(t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} ||DE_{\sigma_k}(\phi_t^k(u))||^2 dt\right)^{\frac{1}{2}} \\
&\le 2(t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} (-\frac{d}{dt}(E_{\sigma_k}(\phi_t^k(u))))dt\right)^{\frac{1}{2}} \\
&\le 2(t_2 - t_1)^{\frac{1}{2}} \left(E_{\sigma_k}(\phi_{t_1}^k(u)) - E_{\sigma_k}(\phi_{t_2}^k(u))\right)^{\frac{1}{2}}
\end{aligned}$$

Thus assuming $t_j \to T_k(u) < \infty$, then $\phi^k_{t_j}(u)$ is Cauchy and converges to a point satisfying equation (6.2) and (6.3), and by assumption is not a critical point of E_{σ_k}, thus contradicting the maximality of $T_k(u)$.

However, since as $E_\sigma(\phi^k_t(u))$ approaches the level $\beta(\sigma) - \epsilon(\sigma_k - \sigma)$, we have

$$\frac{dE_\sigma(\phi^k_t(u))}{dt} \to 0$$

The speed also depends on the cut-off function, and we don't know if it will actually reaches the value $\beta(\sigma) - \epsilon(\sigma_k - \sigma)$.

But we can indeed show that it's not possible to find a positive number ε such that as $t \to T_k(u)$,

$$\inf_{t<T_k(u)} E_\sigma(\phi^k_t(u)) - \beta(\sigma) + \epsilon(\sigma_k - \sigma) = \varepsilon > 0$$

In particular, there will be a time so that,

$$E_\sigma(\phi^k_t(u)) - \beta(\sigma) + \frac{1}{2}\epsilon(\sigma_k - \sigma) < 0$$

Thus giving the same kind of contradiction as above.

Indeed if such a ε exists, then for all $t < T_k(u)$, using φ is increasing and stricly positive on $\mathbb{R}_{>0}$,

$$\frac{dE_\sigma(\phi^k_t(u))}{dt} \leq -\varphi_k(\phi^k_t(u))\frac{\delta^2}{2} \leq -\frac{\delta^2}{2}\varphi\left(\frac{\varepsilon}{\epsilon(\sigma_k - \sigma)}\right) < 0$$

This is a contradiction to the definition of ε, since the flow exists for all time. □

Now we are ready to move on to the last step.

Theorem 6.3. *There is a sequence $\sigma_n \to 0$ and $u_n \in W^{2,2}(S^3, S^2)$, such that*

$$\|DE_{\sigma_n}(u_n)\| = 0, \quad E_{\sigma_n}(u_n) = \beta(\sigma_n), \quad (\partial_\sigma E_\sigma(u_n))_{\sigma=\sigma_n} = o(\frac{1}{\sigma_n \log(\frac{1}{\sigma_n})})$$

We can prove that Indeed, due to Step 1 we can take a sequence of $\sigma_n \to 0$ such that β is differentiable at σ_n and,

$$\beta'(\sigma_n) = o(\frac{1}{\sigma_n \log(\frac{1}{\sigma_n})})$$

Now according to the previous lemma, we can find a sequence of $\sigma_{n,k} \to \sigma_n^+$ and $u_{n,k}$ such that $\|DE_{\sigma_{n,k}}(u_{n,k})\| \to 0$, as $k \to \infty$ and satisfying equation (6.2) and (6.3). Since the functional is Palais-Smale, we have (up to a subsequence) $u_{n,k} \to u_n$ and $\|DE_{\sigma_n}(u_n)\| = 0$. Equation (6.2) to (6.4) follow from the continuity of E_σ and $\partial_\sigma E_\sigma$. Now one can choose $\epsilon_n = o(\frac{1}{\sigma_n \log(\frac{1}{\sigma_n})})$, which gives equation (6.6).

7. Passing to the Limit

Theorem 7.1. *Given the admissible family $\mathcal{A}$ built in section 5, assume $(u_n)_{n\in\mathbb{N}} \in W^{2,2}(S^3, S^2)$ are critical points of E_{σ_n} for σ_n as in section 6, such that*

$$\lim_{\sigma_n\to 0} \sigma_n^2 \int_{S^3} |\Delta u_n|^2 \to 0$$

also,

$$E_{\sigma_n}(u_n) = \inf_{u\in\mathcal{A}} \sup_{x\in\bar{B}^4} E_{\sigma_n}(u) = \beta_{\sigma_n} \to \beta_0$$

we can find a subsequence converging weakly in $W^{1,2}$ to a weakly harmonic map u from S^3 to S^2.

Proof. We have the following by assumption,

$$DE_{\sigma_n}(u_n) = 0$$

$$E_{\sigma_n}(u_n) = \beta_{\sigma_n} \to \beta_0$$

$$\|\sigma_n \Delta u_n\|_{L^2} \to 0 \text{ and } \|\sigma_n \nabla u_n\|_{L^4} \to 0$$

We can first apply Eberlein Šmulian to get a weakly convergent subquence $u_n \in W^{1,2}(S^3, S^2)$, and strong $L^p(p < 6)$ follows from Sobolev Embedding.

We start with the following preparation argument.

The first condition tells us that for all $v \in W^{2,2}(S^3, S^2)$,

$$0 = \int_{S^3} \langle \nabla u_n, \nabla v\rangle - |\nabla u_n|^2 \langle u_n, v\rangle$$
$$+ \sigma_n^2(\langle \Delta u_n, \Delta v\rangle - |\Delta u_n|^2\langle u_n, v\rangle - \langle \Delta u_n, u_n\rangle \Delta\langle u_n, v\rangle - 2\langle \Delta u_n, \langle \nabla u_n, \nabla\langle u_n, v\rangle\rangle_g\rangle)$$

Notice that all the viscosity terms goes to zero uniformly for all $\|v\|_{W^{2,2}(S^3,\mathbb{R}^3)} \leq 1$ due to the third condition, that is,

$$\int_{S^3} \sigma_n^2\langle \Delta u_n, \Delta v\rangle - \sigma_n^2 |\Delta u_n|^2 \langle u_n, v\rangle$$
$$- \sigma_n^2\langle \Delta u_n, u_n\rangle \Delta\langle u_n, v\rangle - 2\sigma_n^2\langle \Delta u_n, \langle \nabla u_n, \nabla\langle u_n, v\rangle\rangle_g\rangle$$
$$\leq \sigma_n^2 \|\Delta u_n\|_{L^2} \|\Delta v\|_{L^2} + \sigma_n^2 \|\Delta u_n\|_{L^2}^2 \|\Delta v\|_{L^\infty}$$
$$+ \sigma_n^2 \|\Delta u_n\|_{L^2}(\|\Delta u_n\|_{L^2}\|v\|_{L^\infty} + \|\Delta v\|_{L^2} + \|\nabla u_n\|_{L^4}\|\nabla v\|_{L^4})$$
$$+ \sigma_n^2 \|\Delta u_n\|_{L^2}\|\nabla u_n\|_{L^4}(\|\nabla u_n\|_{L^4} + \|\nabla v\|_{L^4})$$
$$\to 0$$

The explicit structure of the sphere as the target manifold allows us to derive the Euler Lagrange equation for the weak limit. The following argument is from Evans' lecture notes [8]. For any $\varphi \in C_0^\infty(S^3, \mathbb{R}^3)$,

$$\langle \nabla u_n^i, \nabla \varphi^i \rangle = |\nabla u_n|^2 \langle u_n^i, \varphi^i \rangle + o(1)$$
$$\langle \nabla u_n^j, \nabla \varphi^j \rangle = |\nabla u_n|^2 \langle u_n^j, \varphi^j \rangle + o(1)$$

Thus we plug in $\varphi^i = u_n^j w$ for the first equation and $\varphi^j = u_n^i w$ for the second, for an arbitrary $w(x) \in C_0^\infty(S^3, \mathbb{R})$,

$$o(1) = \langle \nabla u_n^i, \nabla w \rangle u_n^j - \langle \nabla u_n^j, \nabla w \rangle u_n^i$$

Passing to the weak limit we have,

$$\begin{aligned} 0 =& \langle \nabla u^i, \nabla w \rangle u^j - \langle \nabla u^j, \nabla w \rangle u^i \\ &+ \lim_{n\to\infty} \langle \nabla (u_n^i - u^i), \nabla w \rangle u^j - \langle \nabla (u_n^j - u^j), \nabla w \rangle u^i \\ &+ \lim_{n\to\infty} \langle \nabla u_n^i, \nabla w \rangle (u_n^j - u^j) - \langle \nabla u_n^j, \nabla w \rangle (u_n^i - u^i) \\ =& \langle \nabla u^i, \nabla w \rangle u^j - \langle \nabla u^j, \nabla w \rangle u^i \end{aligned}$$

Now we assign $w = u^j \phi^i$, together we have,

$$\begin{aligned} 0 =& \sum_{i,j} \langle \nabla u^i, \phi^i \nabla u^j + u^j \nabla \phi^i \rangle u^j - \langle \nabla u^j, \phi^i \nabla u^j + u^j \nabla \phi^i \rangle u^i \\ =& \sum_{i,j} (u^j)^2 \langle \nabla u^i, \nabla \phi^i \rangle - |\nabla u^j|^2 u^i \phi^i \\ =& \langle \nabla u, \nabla \phi \rangle - |\nabla u|^2 u \cdot \phi \end{aligned}$$

Thus we have solved the Euler Lagrange equation for u.

Since $u_n \in W^{2,2}(S^3, S^2)$ and bounded in $W^{1,2}(S^3, \mathbb{R}^3)$, we may apply strong L^2 convergence and Egorov's theorem to get uniform convergence on E_δ, and $\mathcal{H}^3(E_\delta) < \delta$. Passing $\delta \to 0$, we get $u \in W^{1,2}(S^3, S^2)$, and that u is continuous almost everywhere. Together with the Euler Lagrange equation above, we know that u is weakly harmonic. □

Now we follow the same idea given as in the above proof and give the proof we mentioned in Remark 4.2.

First notice that,

$$W^{-2,2}(S^3, \mathbb{R}^3) \ni DE_\sigma(u_n) = \Delta(u_n - \sigma^2 \Delta u_n) \to 0, \quad u_n \rightharpoonup u \in W^{2,2}(S^3, S^2)$$

Thus one can use a generalized proposition from Functional Analysis (see Brezis' book [4], Chapter 3): if $x_n \rightharpoonup x$ in a Banach space E, and $f_n \to f$ in E^* (the dual space of E), then $\langle f_n, x_n\rangle \to \langle f, x\rangle$ as real numbers. This gives,

$$\int_{S^3} u_n \times \Delta(u_n - \sigma^2 \Delta u_n) \to 0$$

where $\times$ is the cross product in $\mathbb{R}^3$.

Lemma 7.2. *Assume $u_n \in W^{2,2}(S^3, S^2)$ converges weakly to u in the $W^{2,2}$ norm, with*

$$W^{-2,2}(S^3, \mathbb{R}^3) \ni DE_\sigma(u_n) = \Delta(u_n - \sigma^2 \Delta u_n), \quad u_n \times \Delta(u_n - \sigma^2 \Delta u_n) \to 0$$

then u solves the following equation in the distributional sense,

$$u \times \Delta(u - \sigma^2 \Delta u) = 0$$

Proof. Extracting a subsequence we assume that $u_n \to u$ in $W^{1,4}$ and C^0. We write out the equation in divergence form,

$$\begin{aligned} & u_n \times \Delta(u_n - \sigma^2 \Delta u_n) \\ =& \nabla \cdot (u_n \times (\nabla u_n - \sigma^2 \Delta u_n)) - \nabla u_n \times \nabla(u_n - \sigma^2 \Delta u_n) \\ =& \nabla \cdot (u_n \times \nabla(u_n - \sigma^2 \Delta u_n)) + \sigma^2 \nabla u_n \times \nabla \Delta u_n \\ =& \nabla \cdot (u_n \times \nabla(u_n - \sigma^2 \Delta u_n)) + \sigma^2 \nabla \cdot (\nabla u_n \times \Delta u_n) \\ =& \nabla \cdot (u_n \times \nabla(u_n - \sigma^2 \Delta u_n) + \sigma^2 \nabla u_n \times \Delta u_n) \end{aligned}$$

Here one can apply the weak $W^{-1,2}$ convergence of $u_n - \sigma^2 \Delta u_n$, strong $W^{1,4}$ convergence of u_n, and again the proposition combining weak and strong convergence to get the equation for u. Working backwards we get the equation solved by u in the distributional sense,

$$\begin{aligned} 0 =& \nabla \cdot (u \times \nabla(u - \sigma^2 \Delta u) + \sigma^2 \nabla u \times \Delta u) \\ =& u \times \Delta(u - \sigma^2 \Delta u) \end{aligned}$$

On the other hand, we can also write out the equation more explicitly using the cross product in $\mathbb{R}^3$,

$$
\begin{aligned}
&-\int_{S^3} \phi\nabla\cdot(u_n\times\nabla(u_n-\sigma^2\Delta u_n)+\sigma^2\nabla u_n\times\Delta u_n)\\
=&\int_{S^3} -\langle\nabla\phi,\nabla u_n\times(u_n-\sigma^2\Delta u_n)\rangle-\Delta\phi(u_n\times(u_n-\sigma^2\Delta u_n))+\sigma^2\langle\nabla\phi,\nabla u_n\times\Delta u_n\rangle\\
=&\int_{S^3} 2\sigma^2\langle\nabla\phi,\nabla u_n\times\Delta u_n\rangle-\langle\nabla\phi,\nabla u_n\times u_n\rangle+\sigma^2\Delta\phi(u_n\times\Delta u_n)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{S^3} 2\sigma^2\langle\nabla\phi,\nabla u_n\times\Delta u_n\rangle-\langle\nabla\phi,\nabla u_n\times u_n\rangle+\sigma^2\Delta\phi(u_n\times\Delta u_n)\\
&+\int_{S^3} -2\sigma^2\langle\nabla\phi,\nabla u\times\Delta u\rangle+\langle\nabla\phi,\nabla u\times u\rangle-\sigma^2\Delta\phi(u\times\Delta u)\\
=&\int_{S^3} 2\sigma^2\langle\nabla\phi,\nabla(u_n-u)\times\Delta u_n\rangle+2\sigma^2\langle\nabla\phi,\nabla u\times\Delta(u_n-u)\rangle\\
&+\int_{S^3}\langle\nabla\phi,\nabla(u-u_n)\times u\rangle+\langle\nabla\phi,\nabla u_n\times(u-u_n)\rangle\\
&+\sigma^2\int_{S^3}\Delta\phi((u_n-u)\times\Delta u_n)+\Delta\phi(u\times\Delta(u_n-u))\\
\leq& C(\sigma)(\|\nabla\phi\|_{L^4}\|\nabla u-u_n\|_{L^4}\|\Delta u_n\|_{L^2}+\|\nabla\phi\|_{L^2}\|\nabla u-u_n\|_{L^2}\\
&+\|\nabla\phi\|_{L^2}\|\nabla u_n\|_{L^2}\|u-u_n\|_{L^\infty}+\|\Delta\phi\|_{L^2}\|\Delta u_n\|_{L^2}\|u-u_n\|_{L^\infty})\\
&+\sigma^2\int_{S^3} 2\langle\nabla\phi,\nabla u\times\Delta(u_n-u)\rangle+\Delta\phi(u\times\Delta(u_n-u))\\
&\to 0\quad \text{as } n\to\infty
\end{aligned}
$$

Where the last two terms converges follows from integration by parts (one may also just apply the definition of weak convergence).

Hence we have the equation for u,

$$
\int_{S^3} -2\sigma^2\langle\nabla\phi,\nabla u\times\Delta u\rangle+\langle\nabla\phi,\nabla u\times u\rangle-\sigma^2\Delta\phi(u\times\Delta u)=0
$$

Again working backwards we get the equation solved by u in the distributional sense,

$$
\begin{aligned}
0=&\nabla\cdot(u\times\nabla(u-\sigma^2\Delta u)+\sigma^2\nabla u\times\Delta u)\\
=&u\times\Delta(u-\sigma^2\Delta u)
\end{aligned}
$$

This is a more direct way of proving the “convergence” of the Euler Lagrange equation. □

8. Conclusion and Questions

Through the previous sections, we have used MinMax to build a weakly harmonic map from S^3 to S^2, while there are still a lot of questions behind. The author would like to understand these questions in later work.

The first question one may ask is, how strong the convergence in Theorem 7.1 could be. If one manages to show that the convergence would be strong $W^{1,2}$, then we know by the "entropy condition" that our limit u has the limit of the width $\beta(\sigma) \to \beta(0)$ (see section 6). However, we still don't know how smooth our limit point is. It may not be $W^{2,2}$ or even continuous (It is well known that a continuous weakly harmonic map is smooth, Moser [20] or Hèlein [13]). So we don't know if the limit point actually realizes the width $\beta(0)$ for the previously defined $\mathcal{A}$.

At the same time, the width when $\sigma = 0$ already makes sense for $W^{1,2}$ functions. Therefore, as we mentioned in the introduction, one could also have the following family,

$$\mathcal{A}^0 = \Big\{u \in C^0(\overline{B^4}, W^{1,2}(S^3, S^2)) \Big| \max_{\bar{x}\in\partial B^4} \int_{S^3} |\nabla u(\bar{x},\cdot)|^2 \leq \frac{1}{2} C_{S^3}, \\ \frac{\overline{u(\bar{x},\cdot)}}{|\overline{u(\bar{x},\cdot)}|} \text{ is not nullhomotopic from } S^3 \text{ to } S^2 \Big\}$$

But notice that again $W^{1,2}(S^3, S^2)$ is not a Banach manifold. Therefore one cannot apply MinMax on it, or talk about further regularity of a critical point. However, the notion of width still makes sense,

$$\bar{\beta}(0) = \inf_{u\in\mathcal{A}^0} \sup_{x\in\overline{B^4}} \int_{S^3} |\nabla u|^2$$

And the proof in Lemma 5.3 still stands, using $\emptyset \neq \mathcal{A} \subset \mathcal{A}^0$, we get,

$$0 < C_{S_3} < \bar{\beta}(0) \leq \beta(0)$$

One would want to know if equality could be obtained on the right, or at least in cases where one can be more restrictive about the regularity in the family $\mathcal{A}^0$.

Now back to the problem of regularity of our limit u. A classical technique for (partial) regularity of harmonic maps originates from the paper of Sacks and Uhlenbeck [27]. In the specific case of our perturbation, Lamm has shown a result of ε-regularity independent of σ if the domain has dimension 2 [14].

Theorem 8.1. *Assume M, N are closed Riemannian manifolds with M two-dimensional, then there is a $\varepsilon > 0$ and $C > 0$ such that for any small $\sigma > 0$ and any critical point solving*

(3.2), $u \in C^\infty(M,N)$, if

$$E_\sigma(u, B_{32R}(x_0)) = \int_{B_{32R}(x_0)} |\nabla u|^2 + \sigma^2|\Delta u|^2 < \varepsilon \tag{8.1}$$

then for all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} R^i \|D^i u\|_{L^\infty(B_R(x_0))} \leq C\sqrt{E_\sigma(u, B_{32R}(x_0))} \tag{8.2}$$

Furthermore, by a covering argument, the following set has finite counting measure,

$$\Sigma = \cap_{r>0}\{x \in M, \limsup_{k\to\infty} E_{\sigma_k}(u_{\sigma_k}, B_r(x)) \geq \varepsilon\}$$

where for example, the u_{σ_k} could be the choice of critical points as in the end of section 6 for the entropy condition in this paper.

Thus applying the theorem of Arzelà-Ascoli, in dimension 2, one can get strong $W^{1,2}$ and C^k (for all $k > 0$) convergence of the limit outside arbitrary neighborhoods of finitely many points, and there one can apply a blow-up argument.

However, the proof of Lamm is still a two-dimensional result, since a few inequalities used there are no longer valid for higher dimensions (including the Sobolev embedding of $W^{1,1}$ into L^2), and the monotonicity formula is trivial in dimension 2. In dimension 3, this is not the case any more.

We calculate the monotonicity formula below for the case when the domain is flat and the target is a sphere. Since our critical points of E_σ are smooth, we can calculate the monotonicity formula directly. Then we have the following,

$$\sum_i \int_{B_r} x_i \frac{\partial u}{\partial x_i}(\Delta u - \sigma^2 \Delta^2 u)dx = 0$$

Now for the first part we have,

$$\begin{aligned}
&\sum_{i,k} \int_{B_r} x_i \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial^2 x_k} dx \\
&= \sum_{i,k} \int_{\partial B_r} x_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \nu_k ds - \int_{B_r} \delta_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} + \frac{x_i}{2} \frac{\partial}{\partial x_i} \left(\left\| \frac{\partial u}{\partial x_k} \right\|^2 \right) dx \\
&= r \int_{\partial B_r} \left\| \frac{\partial u}{\partial \nu} \right\|^2 ds + \int_{B_r} \frac{|\nabla u|^2}{2} dx - r \int_{\partial B_r} \frac{|\nabla u|^2}{2} ds
\end{aligned}$$

where ν is the unit normal at the sphere, $\nu = \frac{x}{|x|}$. For the second part we have,

$$\begin{aligned}
&\sum_{i,k} - \int_{B_r} x_i \frac{\partial u}{\partial x_i} \Delta^2 u dx \\
=& \sum_{i,k} \int_{B_r} \delta_{ik} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i}(\Delta u) + x_i \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial}{\partial x_k}(\Delta u) - \int_{\partial B_r} r \frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu}(\Delta u) \\
=& - \int_{\partial B_r} r \frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu}(\Delta u) + \int_{\partial B_r} \frac{\partial u}{\partial \nu} \Delta u - \int_{B_r} |\Delta u|^2 \\
&+ \sum_{i,k} \left(\int_{\partial B_r} \frac{x_k x_i}{r} \frac{\partial^2 u}{\partial x_i \partial x_k} \Delta u - \int_{B_r} \delta_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} \Delta u - \sum_i \int_{B_r} x_i \frac{\partial}{\partial x_i} (\frac{|\Delta u|^2}{2}) \right) \\
=& - \int_{\partial B_r} r \frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu}(\Delta u) + \int_{\partial B_r} \frac{\partial u}{\partial \nu} \Delta u + \sum_{i,k} \int_{\partial B_r} \frac{x_k x_i}{r} \frac{\partial^2 u}{\partial x_i \partial x_k} \Delta u - \int_{\partial B_r} \frac{r}{2} |\Delta u|^2 - \int_{B_r} \frac{|\Delta u|^2}{2}
\end{aligned}$$

Putting together one has,

$$\begin{aligned}
0 =& \int_{\partial B_r} r \left| \frac{\partial u}{\partial \nu} \right|^2 + \int_{B_r} \frac{|\nabla u|^2}{2} - \int_{\partial B_r} r \frac{|\nabla u|^2}{2} \\
&+ \sigma^2 \int_{\partial B_r} -r \frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu}(\Delta u) + \frac{\partial u}{\partial \nu} \Delta u + \sum_{i,k} \frac{x_k x_i}{r} \frac{\partial^2 u}{\partial x_i \partial x_k} \Delta u \\
&- \frac{\sigma^2}{2} \int_{B_r} |\Delta u|^2 - \frac{r\sigma^2}{2} \int_{\partial B_r} |\Delta u|^2
\end{aligned}$$

Thus we have the "almost" monotonicity formula in dimension 3,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dr} [\frac{1}{r} \int_{B_r} |\nabla u|^2 + \sigma^2 |\Delta u|^2] =& \frac{1}{r} \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{\sigma^2}{r^2} \int_{B_r} |\Delta u|^2 + \frac{\sigma^2}{r^2} \int_{\partial B_r} \frac{\partial u}{\partial \nu} \Delta u \\
&+ \sigma^2 \sum_{i,k} \int_{\partial B_r} \frac{x_k x_i}{r^3} \frac{\partial^2 u}{\partial x_i \partial x_k} \Delta u - \frac{\sigma^2}{r} \int_{\partial B_r} \frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu}(\Delta u)
\end{aligned}$$

One may look into the right hand side and hope to bound the terms with σ, but the author still don't know if the monotonicity formula will hold.

If one can show that the width is realized by the limit u via for example strong $W^{1,2}$ convergence, then this would be related to a problem conjectured by Rivière.

We first note that as in Smith's paper [28], the Hessian for a smooth harmonic map $f : M \to N$ is given by,

$$H(v, w) = \int_M \langle \nabla_f v, \nabla_f w \rangle - \langle R_N(df, v) df, w \rangle$$

with $v, w \in C^\infty(M, f^{-1}TN)$ and ∇_f the corresponding connection induced by f, R_N the Riemannian curvature tensor on N.

Writing $\rho(v) = R_N(df, v)df$ and using integration by parts, one gets the Jacobi operator,

$$J_f(v) = -\Delta_f v - \rho(v)$$

Using these, it also has been computed that the Hopf map has index 4, see Urakawa [31] or Loubeau-Oniciuc [16]. Now we can state the following conjecture.

Conjecture 8.2. *The only smooth harmonic map from S^3 to S^2 with Morse Index less or equal to 4 are the constant maps and the Hopf map.*

The above conjecture may not be true if one drop the smoothness assumption, or if the harmonic map in consideration cannot be approximated in the $W^{1,2}$ norm via smooth functions into the sphere $C^\infty(S^3, S^2)$.

We also state that there is already a similar result for minimal surfaces index no more than 5, see Urbano [32].

Theorem 8.3. *Let M be a compact orientable nontotally geodesic minimal surface in S^3 (the unit sphere). Then the index of M is at least 5. It's exactly 5 if and only if M is the Clifford torus.*

Applying Michelat [19], each of our critical point u_n to E_{σ_n} has Morse Index no more than 4. Under strong convergence results, one would also want to show that the limit u realizing the width $\beta(0)$ is smooth and has Morse Index no more than 4.

In total, if the positive width is realized by a smooth limit u with index no more than 4, then it will be the Hopf map h as mentioned in the introduction, under the conjecture. We have,

$$\int_{S^3} |\nabla h|^2 \stackrel{?}{=} \beta(0) = \inf_{u \in \mathcal{A}} \sup_{x \in \overline{B^4}} \int_{S^3} |\nabla u(\cdot, x)|^2$$

This will then connect with the 1998 paper by Rivière [25]. One first has the following theorem (the notion of symmetric fibrations are defined in[25]),

Theorem 8.4. *The Hopf fibration minimizes the 3-energy among all of the symmetric fibrations.*

Now if one take a family of conformal diffeomorphism on S^3 parameterized in $\overline{B^4}$, for instance, φ_a as in section 5, and compose it with any $W^{1,3}(S^3, S^2)$ map that is not null-homotopic. Then,

$$\left|S^3\right|^{\frac{1}{3}} \left(\int_{S^3} |\nabla h|^3 \right)^{\frac{2}{3}} \stackrel{?}{=} \beta(0) \le \sup_{a \in \overline{B^4}} E(u \circ \varphi_a) \le \left|S^3\right|^{\frac{1}{3}} \left(\int_{S^3} |\nabla u|^3 \right)^{\frac{2}{3}}$$

where the first equality is true due to the symmetry of the Hopf fibration, $\int_{S^3} |\nabla h|^p = |\nabla h(\cdot)|^p |S^3|$, and the last inequality is due to Hölder inequality and that the 3-energy is conformally invariant.

This shows that the Hopf map minimizes the 3-energy among all $W^{1,3}(S^3, S^2)$ maps that is not null-homotopic, as conjectured by Rivière.

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