CAPILLARY HYPERSURFACES AND VARIATIONAL METHODS IN POSITIVELY CURVED MANIFOLDS WITH BOUNDARY

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Abstract

We study free boundary and capillary minimal hypersurfaces from the variational point of view– they are critical point to the area functional with certain prescribed boundary condition. In particular, we study the interaction of these objects with scalar curvature and boundary convexity.

We first apply the method of generalized soap bubbles (μ-bubbles) to study manifolds with positive scalar curvature; we prove a rigidity result for free boundary minimal hypersurfaces in a 4-manifolds with certain positivity assumptions on curvature. Then we define generalized capillary surfaces (θ-bubbles) and use θ-bubbles to obtain geometric estimates on manifolds with non-negative scalar curvature and uniformly mean convex boundary, including a 1-Urysohn width bound and bandwidth estimate for such 3-manifolds. Lastly, the method of θ-bubble allows us to swap the assumption of positive scalar curvature when using the μ-bubble method with the assumption of positive mean curvature of the boundary, obtaining analogous rigidity results for free boundary minimal hypersurfaces.

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Background and Preliminaries

1.1 Introduction and Main Results

One of the earliest problem in minimal surface theory is the “Plateau’s Problem”, to find existence of a surface of disc type that minimize area while fixing the boundary contour in $\mathbb{R}^3$. The problem was raised in 1760s by Lagrange who also discovered the “Minimal Surface Equation” for a graph in $\mathbb{R}^3$ (see (1.2.1)).

The Plateau’s Problem was named after Joseph Plateau, a Belgian physicist and mathematician who conducted numerous experiments using of “soap bubbles” to study capillary action and surface tension.

In the 1940s, Courant posed the problem of finding area minimizing surfaces whose boundary is constrained in some fixed submanifold of $\mathbb{R}^m$ ([16],[21]).

However, area minimizers do not always exist given a prescribed boundary constraint, or they must satisfy certain rigidity conditions, as we shall also see from results of this thesis. Mathematicians are interested more generally in, critical points to the area functional, while giving such boundary constraint. Such surfaces are called “free boundary minimal surfaces”, meaning as opposed to the Plateau’s problem where the boundary is fixed, now the boundary is allowed to move “freely” as long as remaining constrained in a fixed submanifold.

Free boundary minimal surfaces meet the constraining submanifold orthogonally, and is a special case of “capillary surfaces”, which meet the constraining submanifold at a constant angle, examples of capillary surfaces are the totally geodesic discs in $\mathbb{B}^n \subset \mathbb{R}^n (n \geq 1)$ and the spherical caps obtained by slicing the unit ball with a hyperplane. Both examples are capillary stable ([52]), meaning the second variation is non-negative with respect to some admissible variations (see section 1.2).

In this thesis we are focused on two problems with intertwining interest:

- rigidity results of free boundary minimal hypersurfaces in positively curved manifolds with boundary;

- geometric estimates for manifolds with non-negative scalar curvature and uniformly mean convex boundary, using generalized stable capillary surfaces, called "θ-bubbles" (see chapter 4).

This thesis is organized as follows.

In section 1.2, we begin with some preliminaries that extend the notion of smooth minimal surfaces to sets with finite perimeters that are critical points to some generalized area functional, we compute the first and second variations and introduce the Jacobi operator.

In section 1.3, we define "μ-bubbles", introduced by Gromov ([24]), and briefly summarize some recent progress on scalar curvature using the μ-bubble method.

In particular, Schoen and Yau's proof of positive mass theorem ([56]) presented the strength of minimal surface theory in application to the study of scalar curvature and general relativity. The idea of "conformal descent" by Schoen and Yau can be generalized to μ-bubbles, and lead to resolution of important questions in geometric analysis recently ([12], [13], [14], [29], [26], [69], [70], [68]).

In chapter 2, we prove rigidity results for stable minimal hypersurface in a compact 4-manifold with non-negative 2-intermediate Ricci curvature, positive scalar curvature and weakly convex boundary (see section 2.1). An example of such a 4-manifold is a weakly convex spherical cap. The result also holds for non-compact 4-manifolds if we assume uniformly positive scalar curvature, and with weakly bounded geometry. This extends the results of Chodosh, Li and Stryker ([15]) using the μ-bubble method for manifolds with no boundary. The main ingredients for the free boundary case is, the use of weakly bounded geometry assumption to prove curvature estimates for stable free boundary minimal hypersurfaces and a volume bound for arbitrary balls with fixed radius (see section 2.3); the use of intermediate Ricci curvature and 2-convexity assumptions to prove a Liouville theorem for stable free boundary minimal hypersurfaces (see section 2.5); and the use of free boundary μ-bubbles (see section 2.6) to prove the almost linear volume growth as of [15] for manifolds with no boundary.

In chapter 3, we will use generalized capillary surfaces to study the geometry of 3-manifolds with non-negative scalar curvature and uniformly mean convex boundary (see section 3.1). We will prove that such 2-manifolds are discs with bounded circumference and diameter (see section 3.2); the boundary of such simply connected 3-manifolds must have uniformly bounded 1-Urysohn width (see section 3.3); and we will prove a sharp bandwidth estimate for bands over a surface with non-positive Euler characteristic (see section 3.4).

In chapter 4, we prove rigidity results for stable free boundary minimal hypersurfaces in a 4-manifold with the same assumption as in chapter 2, except now we don't require the positivity of scalar curvature, instead we assume uniformly positive mean curvature of the boundary (see section 4.1). Instead of using the method of μ-bubble tailored for manifolds with positive scalar curvature, we use the tools we developed in chapter 3, the method of "θ-bubble" tailored for manifolds with non-negative scalar curvature and uniformly mean convex boundary. We observe that such manifolds also have the phenomenon of "conformal descent" and the tools we used for 3-manifolds applies to general dimensions.

The main difference to chapter 2 is now that the method of θ-bubble can only control the 1-Urysohn width of the boundary of such simply connected 3-manifolds, which is not enough to obtain intrinsic volume of a free boundary stable minimal hypersurface $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$. We need to further exploit the stability inequality: we will show that each component of ∂M must be non-compact and has an end in the only non-parabolic end of M. This allows us to exhaust the only non-parabolic end using θ-bubbles, thus obtaining the same almost linear volume growth as in chapter 2.

1.2 Preliminaries

In 1762, Lagrange found the Euler-Lagrange equation for a graph $z = z(x, y)$ in $\mathbb{R}^3$ that minimizes area on any compact set,

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = 0 \tag{1.2.1}$$

where $z : D \to \mathbb{R}$ is a smooth function over a open smooth domain $D \subset \mathbb{R}^2$.

A graph satisfying the above equation (called "the Minimal Surface Equation") is a critical point to the area functional.

Lagrange found one solution, the plane. More and more interesting solutions were found in the 19th century and afterwards.

The Plateau's problem of finding an area-minimizing disc-type surface realized by a smooth graph $f \in C^\infty(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$, while $f|_{\partial\mathbb{D}}$ is a weakly monotone parametrization of a fixed smooth simple closed curve, was complete solved by Douglas and Radó independently in the 1930s'.

Generalization of minimal surface theory to higher dimensions and general Riemannian manifolds have been intensively studied. We are interested in the variational viewpoint of the minimal surface theory.

Definition 1.2.1 (Admissible Variations)**.** *Consider a smooth immersions $F : M \hookrightarrow X$ with $\partial M \subset \partial X$ if $\partial X \neq \emptyset$. we say a variation is admissible if it is a family of immersions $F_t : M \times (-1, 1) \hookrightarrow X$ for $t \in (-1, 1)$ that agrees with F at $t = 0$ or outside of any compact set and $F_t(\partial M_t) \subset \partial X_t$ for $t \in (-1, 1)$.*

In this thesis, a minimal hypersurface is always a two-sided immersed hypersurface $M \hookrightarrow X$ that is a critical point to the area functional with respect to all admissible variations. Therefore, a free boundary minimal hypersurface $(M, \partial M)$ in a manifold $(X, \partial X)$ with boundary is a critical point to the area functional among hypersurfaces whose boundary remains in the boundary of the ambient manifold.

Lemma 1.2.2 (First Variation)**.** *We compute the first variation formula for a smooth immersed hypersurface given ν_M, a choice of unit normal vector field along M and*

$Y = \frac{d}{dt}\big|_{t=0} F_t(M)$:

$$\frac{d}{dt}\bigg|_{t=0} Area(M_t) = \int_M \operatorname{div}_M Y^\perp + \int_{\partial M} Y \cdot \nu_{\partial M}$$
$$= -\int_M H \cdot Y + \int_{\partial M} Y \cdot \nu_{\partial M},$$

where H is the mean curvature of $M \hookrightarrow X$ with respect to ν_M.

Throughout the paper we use this convention of the second fundamental form, $\mathrm{II}_M(Y, Z) = -\langle \overline{\nabla}_Z Y, \nu_M \rangle$ given a choice of normal vector field ν_M for a hypersurface and $\overline{\nabla}$ the Levi-Civita connection on the ambient manifold X. Then define mean curvature as $H_M = \operatorname{tr}(\mathrm{II})$. In this convention, mean curvature of a sphere with outward unit normal in the Euclidean space is positive.

By the first variation formula, an immersed submanifold $M \hookrightarrow X$ is a minimal hypersurface if and only if its mean curvature vanishes everywhere. A free boundary minimal hypersurface is equivalently a submanifold with vanishing mean curvature and meets the boundary of the ambient manifold orthogonally (that is, the outward unit normal of ∂M agrees with the outward unit normal of ∂X; so the second fundamental form of $\partial M \hookrightarrow M$ is the same as restriction of the second fundamental form of $\partial X \hookrightarrow X$ on $T\partial M$).

Lemma 1.2.3 (Second Variation)**.** *M is called stable if its second variation is non-negative among such admissible variations.*

Then given ν_M a choice of unit normal vector field along M, we have the following stability inequality for any compactly supported Lipschitz function $\phi = Y \cdot \nu_M$,

$$\int_M |\nabla_M \phi|^2 \geq \int_M (|\mathrm{II}|^2 + \operatorname{Ric}_X(\nu_M, \nu_M))\phi^2 + \int_{\partial M} A(\nu_M, \nu_M)\phi^2,$$

where II is the second fundamental form of $M \hookrightarrow X$ and X the second fundamental form of $\partial X \hookrightarrow X$ and Ric_X stands for the Ricci curvature of X.

We use the second variation of a (free boundary) minimal hypersurface $M \hookrightarrow X$

to define the quadratic form $Q_M(f,f)$:

$$Q_M(f,f) = \int_M -f\Delta_M f - (\mathrm{Ric}_X(\nu_M,\nu_M) + |\mathrm{I\!I}|^2)f^2 + \int_{\partial M} f\partial_{\nu_M} f - A(\nu_M,\nu_M).$$

The associated Jacobi operator of M is defined as,

$$J_M(f) := \Delta_M f + (\mathrm{Ric}_X(\nu_M,\nu_M) + |\mathrm{I\!I}|^2)f.$$

Using the Gauss-Codazzi equation $R_X = R_M + 2\,\mathrm{Ric}_X(\nu_M,\nu_M) + |\mathrm{I\!I}|^2 - H^2$,

$$J_M(f) = \Delta_M f + \frac{1}{2}(R_X - R_M + |\mathrm{I\!I}|^2 + H^2)f.$$

This relates the Jacobi operator of stability inequality to the conformal Laplacian,

$$L_M(f) = \Delta_M f - \frac{n-2}{4(n-1)}R_M f.$$

The technique of using stability inequality of minimal hypersurfaces to study scalar curvature problem is called Schoen-Yau's "conformal descent" (see section 1.3).

As we are interested in positively curved manifolds with boundary. We first introduce more precisely the curvature conditions for ambient manifolds, and adopt the non-standard short phrase "weakly bounded geometry" given in Franz ([21]) to indicate we consider Riemannian manifolds $(M,\partial M,g)$ with the following,

- either the scalar curvature of M is uniformly positive ($R_M \geq R_0 > 0$), and the mean curvature of ∂M is non-negative ($H_{\partial M} \geq 0$) with no minimal components;
- or the scalar curvature of M is non-negative ($R_M \geq 0$), and the mean curvature of ∂M is uniformly positive ($H_{\partial M} \geq H_0 > 0$).

This notion is consider "weakly positive geometry" as compared to the "positive geometry" conditions when the Ricci curvature is assumed to be positive or non-negative.

Examples of such manifolds include the spherical caps (the first case of "weakly positive geometry") and the Euclidean balls (the second case).

Capillary hypersurfaces have proven useful to study comparison theorems in manifolds with non-negative scalar curvature and uniformly mean convex boundary (see for example [40],[9]).

Definition 1.2.4 (Capillary surfaces)**.** *A capillary hypersurface is the boundary of a smooth open set Ω in a compact Riemannian manifold $(N, \partial N)$ that is a critical point to the following functional,*

$$\mathcal{A}_0(\Omega) = Area(\partial\Omega) - \int_{\partial N \cap \Omega} \cos\theta,$$

for a constant $\theta \in (0, \pi)$, and among admissible variations with fixed volume ratio $\lambda_0 = \frac{|\Omega|}{|N|}$.

The first variation formula for capillary surfaces gives an equivalent definition: a capillary surface has constant mean curvature and constant intersection angle with the ambient boundary. In particular, a free boundary minimal surface is a capillary surface.

Lemma 1.2.5 (Second variation, [52])**.** *A stable capillary hypersurface Σ has non-negative second variation among admissible variations:*

$$\begin{aligned} 0 \leq \frac{d}{dt}\bigg|_{t=0} \mathcal{A}_0(\Omega_t) = &\int_\Sigma |\nabla\phi|^2 - (\mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) + |\mathrm{II}_\Sigma|^2)\phi^2 \\ &- \int_{\partial\Sigma} \frac{\phi^2}{\sin\theta}(\mathrm{II}_{\partial N}(\bar\nu, \bar\nu) - \cos\theta \mathrm{II}_\Sigma(\nu, \nu)), \end{aligned}$$

for any $\phi \in C_c^\infty(\Sigma)$; here ν_Σ the outward pointing unit normal of $\Sigma \hookrightarrow \Omega$, II stands for the corresponding second fundamental forms, and $\bar\nu$ (respectively ν) stands for the outward unit normal of $\partial\Sigma \hookrightarrow \Omega \cap \partial N$ (respectively $\partial\Sigma \hookrightarrow \Sigma$).

These definitions and calculations can be generalized to sets with lower regularity.

Definition 1.2.6 (Sets with finite perimeters, [46])**.** *An open set E has locally finite perimeter (also called a "Caccioppoli set") in an open $U \subset \mathbb{R}^n$, if and only if for any $K \subset\subset U$,*

$$\sup\{\int_{K\cap E} \mathrm{div}\,\phi : \phi \in C_c^\infty(K, \mathbb{R}^n), \|\phi\|_\infty \leq 1\} < \infty.$$

An open set E has locally finite perimeter in $V \cap \mathbb{H}_+$ for V an open set in $\mathbb{R}^n$, if and only if E is an open set with finite perimeter in V, and $E \subset \mathbb{H}_+$.

Equivalently, E has locally finite perimeter, if and only if the distributional derivative $\nabla \chi_E$ is a Radon measure μ_E. The first improved regularity we can get from sets with finite perimeters is that the support of μ_E is an $n-1$ rectifiable set called the "reduced boundary".

Definition 1.2.7 (Reduced Boundary, [46]). *The reduced boundary $\partial^* E$ of a set E with locally finite perimeter is,*

$$\left\{ x \in spt(\mu_E) : \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))} = \nu_E(x) \text{ for some } \nu_E(x) \in \mathbb{S}^{n-1} \right\}$$

Theorem 1.2.8 (De Giorgi Structure Theorem, [46]). *If E has locally finite perimeter in U, then $\partial^* E$ is rectifiable and,*

$$\mu_E = \nu_E \mathcal{H}^{n-1}|_{\partial^* E}, \quad |\mu_E| = \mathcal{H}^{n-1}|_{\partial^* E}.$$

And we can integrate by parts in the following sense,

$$\int_E \nabla \phi = \int_{\partial^* E} \phi \cdot \nu_E, \quad \forall \phi \in C_c^\infty(U).$$

Using a partition of unity, one can define Caccioppoli sets in a Riemannian manifold with or without boundary analogously.

1.3 Scalar Curvature and μ-Bubbles

As we mentioned in section 1.2, the Jacobi operator allows us to use stable minimal hypersurfaces to study the geometry of the ambient manifolds, in particular to problems related to scalar curvature.

Schoen and Yau in particular used the method of "conformal descent" of positive scalar curvature metric (PSC) to stable minimal hypersurfaces in a PSC manifold, to prove the following conjecture.

Theorem 1.3.1 (Geroch's Conjecture, [25],[54],[53]). *The connect sum $\mathbb{T}^n \# X$ when X is any closed manifold, has no metric of positive scalar curvature.*

The resolution of Geroch conjecture also leads to important applications in general relativity.

Theorem 1.3.2 (Positive Mass Theorem, [53]). *Let (M^n, g) be an asymptotically flat manifold with $R_g \geq 0$, $3 \leq n \leq 7$, then its ADM mass $m_g \geq 0$, and $m_g = 0$ if and only if M is isometric to the Euclidean space.*

In the proof of Theorem 1.3.1, Schoen and Yau utilized the existence and regularity (in dimension $n \leq 7$) of area minimizing minimal hypersurfaces in a non-trivial homology class.

While it is not always possible to find stable minimal hypersurfaces in an arbitrary manifold (e.g. when the manifold is contractible so the homology classes are trivial), the idea of conformal descent can be applied to a generalized notion of minimal hypersurfaces, called μ-bubbles.

Definition 1.3.3 (Gromov [24], [26]). *A μ-bubble in a Riemannian manifold (M^n, g) is a set of finite perimeter that minimize the following functional,*

$$\mathcal{A}_h(\Omega) = \mathcal{H}^{n-1}(\partial^* \Omega) - \int_M h(\chi_\Omega - \chi_{\Omega_0}),$$

given a smooth function $h : M \to \mathbb{R}$ and a fixed smooth domain $\Omega_0 \subset M$.

We note that in the case $h = 0$, a μ-bubble is area minimizing. Using regularity for almost area minimizing hypersurfaces ([61]), if $3 \leq n \leq 7$, a μ-bubble is a smooth submanifold. The first variation formula of $\mathcal{A}_h(\cdot)$ implies $H_\Sigma = h|_\Sigma$, so $\mathcal{A}_h(\cdot)$ is also called the "prescribed mean curvature" functional.

We have the following second variation formula for μ-bubble,

Lemma 1.3.4 (Second Variation, [13]). *Assume Ω is a smooth μ-bubble in M and denote $\Sigma = \partial^* \Omega$, then given any normal variation generated by $X_t = \phi \nu_\Sigma$, with ν_Σ*

the outward pointing unit normal of $\Sigma \subset \Omega$ *and* $\phi \in C_c^\infty(\Sigma)$,

$$\begin{aligned}
0 \le \frac{d}{dt}\bigg|_{t=0} \mathcal{A}_h(\Omega_t) &= \int_\Sigma |\nabla\phi|^2 - (\mathrm{Ric}_M(\nu_\Sigma, \nu_\Sigma) + |\mathrm{II}_\Sigma|^2)\phi^2 - \phi^2 \nabla_{\nu_\Sigma} h \\
&= \int_\Sigma |\nabla\phi|^2 - \frac{1}{2}(R_M - R_\Sigma + |\mathrm{II}_\Sigma|^2 + H_\Sigma^2)\phi^2 - \phi^2 \nabla_{\nu_\Sigma} h \\
&\le \int_\Sigma |\nabla\phi|^2 - \frac{1}{2}(R_M - R_\Sigma)\phi^2 - \phi^2(\nabla_{\nu_\Sigma} h + \frac{n+1}{n} h^2)
\end{aligned}$$

Using μ-bubbles, Gromov proved the following sharp bandwidth estimate for an over-torical band.

Theorem 1.3.5 (Bandwidth Estimate, [30])**.** *Let* $M \approx \mathbb{T}^n \times [-1,1] (2 \le n \le 6)$ *be a Riemannian manifold with scalar curvature* $R_M \ge n(n+1)$*, then*

$$d_M(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{+1\}) \le \frac{2\pi}{n+1}.$$

We know from Bonnet-Meyers theorem that a surface with uniform PSC has uniformly bounded diameter. What can we say about 3-manifolds with uniformly PSC?

Topologically, we can characterize a compact 3-manifold with uniform PSC using the work of Gromov and Lawson ([25]), the resolution of Poincaré conjecture by Perelman ([49],[50],[51]). For complete non-compact manifolds, characterization results were obtained by Chang, Weinberger and Yu ([10], assuming contractibility) and Bessières, Besson and Maillot ([5], assuming bounded geometry), Wang ([62], removed the assumption of bounded geometry).

Theorem 1.3.6. *A complete 3-manifold with uniformly positive scalar curvature must be (possibly infinitely many) connect sums of* $\mathbb{S}^2 \times \mathbb{S}^1$ *and space forms.*

Quantitatively, 3-manifolds with uniform PSC is close to being 1-dimensional as proven by Liokumovich-Maximo and Liokumovich-Wang.

Theorem 1.3.7 (Waist Inequality)**.** *Let* M *be a complete 3-manifold with* $R_M \ge R_0 > 0$*, then there is a proper Morse function* $f : M \to \mathbb{R}$ *and* $A_0 > 0, d_0 > 0$ *such* *that for any* $t \in \mathbb{R}$*, each connected component* Γ *of* $f^{-1}(t)$ *has,*

$$diam_M(\Gamma) \leq d_0, \quad Area(\Gamma) \leq A_0.$$

Another way to give geometric estimates of PSC 3-manifolds is through Urysohn widths bound.

Definition 1.3.8 (Urysohn Width)**.** *A metric space* (X, d) *has* k*-Urysohn width bounded by* $d_0 > 0$ *if there is a continuous map to a (connected)* k*-dimensional simplecial complex* G*, such that*

$$\operatorname{diam}_d(f^{-1}(g)) \leq d_0, \quad \forall g \in G.$$

Using Bonnet-Meyers, we know that uniformly positive Ricci lower bound implies a uniform upper bound on 0-Urysohn width: the diameter.

Gromov made the following conjecture that uniformly positive scalar curvature gives codimension 2 control of the growth of a Riemannian manifold.

Conjecture 1.3.1 (Gromov [24])**.** If M is a complete n-dimensional Riemannian manifold with $R_M \geq R_0 > 0$, then the $n-2$ Urysohn width of M is bounded from above by $\frac{c_n}{\sqrt{R_0}}$.

For example, consider any closed Riemannian manifolds M^n and the connect sum $M \times \mathbb{S}^2(r)$, if r is sufficiently small then $M \times \mathbb{S}^2(r)$ has uniform PSC and has $n-2$ Urysohn width bound.

The method of μ-bubble allows Chodosh and Li to give a very concise proof on Urysohn widths bound for simply connected 3-manifolds.

Theorem 1.3.9 (Chodosh-Li, [13])**.** *If* (M, g) *is a simply connected 3 manifold with* $R_g \geq 2$*, then the* 1*-Urysohn width of* M *is bounded from above by* 12π*. Furthermore, there is a continuous map* $f : M \to T$ *where* T *is a* 1*-dimensional simplicial complex with no loops (a tree), such that* $\operatorname{diam}_g(f^{-1}(t)) \leq 12\pi$ *for any* $t \in T$*.*

We note that Gromov's conjecture 1.3.1 for manifolds with dimension 4 or higher is still widely open.

The resolution to the 1-Urysohn width bound to non-compact simply connected 3-manifolds has two important consequence. The first one is two directions of generalization of Geroch conjecture.

Definition 1.3.10. *A topological manifold N is aspherical if the one of the following equivalent condition holds,*

- *the universal cover $\tilde{N}$ of N is contractible;*
- *the higher homotopy groups $\pi_k(N)$ for $k \geq 2$ vanishes.*

Theorem 1.3.11 (Chodosh-Li, [13])**.** *Extending Geroch conjecture:*

- *Closed aspherical manifolds of dimension 4 or 5 have no complete smooth metric of positive scalar curvature;*
- *The connect sum of an arbitrary manifold X^n and the torus $\mathbb{T}^n$ does not admit a complete smooth metric of positive scalar curvature.*

We note that in the proof of Schoen and Yau, $\mathbb{T}^n \# X$ has no PSC when X is compact, the authors use "conformal descent" of scalar curvature and take advantage of the abundance of homology in $\mathbb{T}^n$. The torus $\mathbb{T}^n$ is aspherical. While in the proof of Chodosh and Li, the authors pass to the universal cover, which has no topology to make use of. It's the introduction of μ-bubble that allows the idea of "conformal descent" to still be applied in such a situation.

In another direction, Chodosh, Li and Stryker proved that each end in a simply connected 3-manifolds with PSC has linear volume growth, allowing them to obtain rigidity results for stable minimal hypersurfaces in a positively curved 4-manifold.

Theorem 1.3.12 (Chodosh-Li-Stryker, [15])**.** *Consider (X^4, g) with weakly bounded geometry and*

$$\mathrm{Ric}_2^X \geq 0, \quad R_X \geq R_0 > 0.$$

Then any complete two-sided stable minimal hypersurface $M^3 \hookrightarrow X^4$ must have

$$|\mathrm{II}_M| = 0, \quad \mathrm{Ric}(\nu_M, \nu_M) = 0,$$

for ν_M a choice of unit normal along M.

Here the condition $\mathrm{Ric}_2^X \geq 0$ is an intermediate curvature assumption that lies between sectional curvature and Ricci curvature, see chapter 2 for the definition and explanations of why this is a reasonable assumptions for rigidity results of stable minimal hypersurfaces in 4-manifolds.

Chapter 2

PSC 3-Manifolds and Free Boundary μ-Bubbles

This chapter extends the method of Chodosh-Li-Stryker ([15]) to free boundary minimal hypersurfaces in ambient manifolds with boundary.

Precisely, we show that the combination of nonnegative 2-intermediate Ricci Curvature and strict positivity of scalar curvature forces rigidity of two-sided free boundary stable minimal hypersurface in a 4-manifold with bounded geometry and weakly convex boundary.

The results in this chapter come from [65].

2.1 Introduction

Recall the second variation formula in Lemma 1.2.3.

Using the stability inequality, the positivity of the curvature tensor of the ambient manifold X^n has been exploited to obtain rigidity or non-existence results of stable minimal (free boundary) hypersurfaces. When X^n has non-negative Ricci and M is closed, then M must be totally geodesic ([59]) and the Ricci curvature must vanish in the normal direction along M (in particular, strictly positive Ricci curvature implies non-existence of stable minimal hypersurfaces); when X^n has positive scalar curvature (PSC) and M is closed, then Scheon and Yau proved M must also has PSC ([53],[54]).

See [15] for a more complete review of the literature.

While if the ambient manifold is non-compact, to use the same method, we need to bound the volume growth of the minimal hypersurface.

For a surface in a 3-manifold, if the scalar curvature has $R_X \geq 0$ and $\partial X = \emptyset$, then Fischer-Colbrie and Schoen proved M with the induced metric is either conformal to a plane or a cylinder, and the later case implies that M is totally geodesic, intrinsically flat, $R_X|_M = 0$ and $\mathrm{Ric}_X(\nu_M, \nu_M) = 0$ along M ([20]). Furthermore, if $R_X \geq 1$, then M must be compact ([55],[25]), and also admit a metric of PSC. The idea that the positivity of scalar curvature can be "inherited" has seen many fruitful applications ([56], [13]).

In higher dimension, the following two examples in [15] explain analogous results to the case of compact minimal hypersurfaces cannot be obtained via the local variational method:

- there is a stable totally geodesic embedding of $\mathbb{R}^3 \hookrightarrow (\mathbb{R}^4, g)$ where $(\mathbb{R}^4, g)$ has strictly positive sectional curvature everywhere;
- there is a stable totally geodesic embedding of $M^3 \hookrightarrow X^4$ where X has uniformly positive Ricci curvature in a tubular neighborhood of M^3.

This explains the assumption of intermediate Ricci curvature $\mathrm{Ric}_2^X \geq 0$ in Chodosh, Li and Stryker [15]. Apart from the result in [15] using the method of μ-bubble to give an almost-linear volume growth bound for an end of a stable minimal hypersurface in a non-compact 4-manifold as in Theorem 1.3.12, we note that recently Catino-Mastrolia-Roncoroni [8] has given rigidity results of complete stable minimal hypersurfaces in $\mathbb{R}^4$ or a positively curved Riemannian manifold X^n when $n \leq 6$, where the authors look at a suitable positive curvature condition introduced by Shen and Ye ([57]).

For an ambient 4-manifold $(X, \partial X)$, we say X has weakly convex boundary if the second fundamental form of the boundary is positive semi-definite. The so-called non-negative "2-intermediate Ricci curvature" assumption, denoted as $\mathrm{Ric}_2 \geq 0$, lies between non-negative sectional curvature and non-negative Ricci curvature, and will be explained in section 2.2. The author obtained analogous results to [15] for free boundary minimal hypersurfaces.

Theorem 2.1.1 ([65]). *Consider $(X^4, \partial X)$ a complete Riemannian manifold with weakly convex boundary, $R \geq 2$, $\mathrm{Ric}_2 \geq 0$, and weakly bounded geometry. Then any complete stable two-sided immersion of free boundary minimal hypersurface $(M, \partial M) \hookrightarrow (X, \partial X)$ is totally geodesic, $Ric(\eta, \eta) = 0$ along M and $A(\eta, \eta) = 0$ along ∂M, for η a choice of unit normal over M.*

In particular, any compact manifold $(X^4, \partial X)$ with positive sectional curvature and weakly convex boundary will satisfy the assumption above. This gives the following nonexistence result:

Corollary 2.1.2. *There is no complete two-sided stable free boundary minimal immersion in a compact manifold $(X^4, \partial X)$ with positive sectional curvature and weakly convex boundary.*

We will note two aspects that are mainly different from the case without boundary in [15] and require new ingredients.

The first is the notion of parabolicity and non-parabolicity for an end E of manifolds with non-compact boundary, where we need to look at a (weakly) harmonic function f with mixed (Dirichlet-Neumann) boundary conditions on two different parts of the boundary $\partial E = \partial_0 E \cap \partial_1 E$. Standard elliptic regularity tells us that f is smooth away from the points of intersection $\partial_0 E \cap \partial_1 E$. By the work of Miranda [47] we can see that f is continuous (and bounded) around each point of intersection. Then the work of Azzam and Kreyszig [3] gives that if the interior angle of intersection θ is small, then f is $C^{k,\alpha}$ for k and α depending on θ. This allows us to control the number of non-parabolic ends of M.

Theorem 2.1.3. *Let $(X^4, \partial X)$ be a complete manifold with $\mathrm{Ric}_2 \geq 0, A_2 \geq 0$, and $(M, \partial M)$ a free boundary two-sided stable minimal immersion with infinite volume, then for any compact set $K \subset M$, there is at most 1 non-parabolic component in $M \setminus K$.*

Here we write A as the second fundamental form of ∂X in X, then $A_2 \geq 0$ is an intermediate assumption lying between convexity and mean convexity, which will be explained in Section 2.2.

The second ingredient is the bound of volume growth on a ball of fixed radius in M. In [15], since M has no boundary, with a uniform lower Ricci bound, we can obtain volume bound via Bishop-Gromov volume comparison theorem. To apply the same for the free boundary case, one can exploit the assumption that X has convex boundary. On the other hand, we can actually use the weakly bounded geometry assumption (that is already needed if one needs to apply blow-up argument to an arbitrary non-compact Riemann manifold).

Lemma 2.1.4. *Let $(X^n, \partial X, g)$ be a complete Riemannian manifold with weakly bounded geometry at scale Q, and $(M^{n-1}, \partial M) \hookrightarrow (X, \partial X)$ a complete immersed submanifold with uniformly bounded second fundamental form, then the following is true,*

- *there is $0 < N < \infty$ such that for any $p \in M$, the maximum number of disjoint balls of radius δ centered around points in $B^M_{4\delta}(p)$ is bounded by N,*
- *for any $R > 0$, there is a constant $C = C(R, Q)$ such that the volume of balls of radius R around any point in M is bounded by C.*

Proof of the lemma used an inductive covering argument in Bamler-Zhang [4]. Preliminaries and outline of the paper is given in Section 2.2.

2.2 Preliminaries

Recall for a free boundary minimal immersion $(M, \partial M) \hookrightarrow (X, \partial X)$, we write $\mathrm{I\!I}$ for the second fundamental form of $M \hookrightarrow X$ and A for the second fundamental form of $\partial X \hookrightarrow X$.

We now introduce the curvature assumptions we made on the ambient manifolds (see also [15]).

Definition 2.2.1. *We say that X has* $\mathrm{Ric}_2 \geq 0$*, i.e. nonnegative 2-intermediate Ricci curvature , if*

$$R(v,u,u,v) + R(w,u,u,w) \geq 0, \tag{2.2.1}$$

for any $x \in X$ and any orthonormal vectors u, v, w of T_xM, where $R(\cdot,\cdot,\cdot,\cdot)$ represents the Riemann curvature tensor of X.

Remark 2.2.2. *Note since* Ric *is symmetric, as long as the dimension of X is at least 3,* $\mathrm{Ric}_2 \geq 0$ *implies that* $\mathrm{Ric}(u,u) \geq 0$ *for any vector u in the tangent plane of X and so* $\mathrm{Ric} \geq 0$ *everywhere.*

Using $\mathrm{Ric}_2 \geq 0$ of the ambient manifold and Gauss Equation, we can control the Ricci curvature from below by the second fundamental form of a minimal immersion.

Lemma 2.2.3 ([15], Lemma 2.2)**.** *Consider $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ immersed free boundary minimal hypersurface, if X has* $\mathrm{Ric}_2 \geq 0$*, then*

$$\mathrm{Ric}_M \geq -|\mathrm{I\!I}|^2. \tag{2.2.2}$$

Remark 2.2.4. *The proof works in other dimensions too, the same conclusion holds for all X^n with $n \geq 3$. When $n = 3$, we would need X to have positive sectional curvature. If $n \geq 4$, we only need the following weaker assumption named* $\mathrm{Ric}_{n-2} \geq 0$*, meaning for any orthonormal vectors $e_1, ..., e_{n-1}$ at a tangent plane of X, we have*

$$\sum_{k=2}^{n-1} R(e_k, e_1, e_1, e_k) \geq 0.$$

We can in fact get a sharper bound with a constant depending on the dimension.

Lemma 2.2.5 ([15], Lemma 4.2)**.** *Consider $(M^{n-1}, \partial M) \hookrightarrow (X^n, \partial X)$ immersed free boundary minimal hypersurface, if X has* $\mathrm{Ric}_{n-2} \geq 0$*, then*

$$\mathrm{Ric}_M \geq -\frac{n-2}{n-1}|\mathrm{I\!I}|^2. \tag{2.2.3}$$

We also define an analogous "2-convexity" condition for $\partial X \hookrightarrow X$, lying between convexity and mean convexity.

Definition 2.2.6. *For $(X, \partial X)$ a complete manifold with boundary, recall A is the second fundamental form of $\partial X \hookrightarrow X$, we say that $A_2 \geq 0$ if for any orthonormal vectors e_1, e_2 on a tangent plane of ∂X, we have $A(e_1, e_1) + A(e_2, e_2) \geq 0$.*

The condition of "2-convexity" and $\mathrm{Ric}_2^X \geq 0$ will be useful combined with the stability inequality to obtain a Liouville theorem for stable minimal hypersurfaces in X^4, see section 2.5.

Also, to obtain blow up analysis needed for curvature estimates of stable minimal hypersurfaces in an arbitrary ambient Riemannian manifold, we require $(X, \partial X)$ to have weakly bounded geometry, defined as below.

Definition 2.2.7. *We say a complete Riemannian manifold with boundary $(X, \partial X, g)$ has weakly bounded geometry (up to the boundary) at scale Q, if for this $Q > 0$, there is $\alpha \in (0, 1)$ such that for any point $x \in X$,*

- *there is a pointed $C^{2,\alpha}$ local diffeomorphism $\Phi : (B_{Q^{-1}}(a), a) \cap \mathbb{H}_+ \to (U, x) \subset X$, for some point $a \in \mathbb{R}^n$, here $\mathbb{H}_+$ is the upper half space in $\mathbb{R}^n$;*
- *and if $\partial X \cap U \neq \emptyset$, then $\Phi^{-1}(\partial X \cap U) \subset \partial \mathbb{H}_+$.*

Furthermore, the map Φ has,

- *$e^{-2Q} g_0 \leq \Phi^\star g \leq e^{2Q} g_0$ as two forms, with g_0 the standard Euclidean metric;*
- *$\|\partial_k \Phi^\star g_{ij}\|_{C^\alpha} \leq Q$, where i, j, k stands for indices in Euclidean space.*

We will prove two consequences of this condition in the next section: one is the curvature estimates for stable free boundary minimal hypersurface following the resolution of stable Bernstein theorem of Chodosh and Li [14],[12]– any two-sided complete stable minimal hypersurface in $\mathbb{R}^4$ is flat; the other is a volume control of balls of fixed radius by a constant depending on the coefficient Q in the definition above.

Until now we don't really need to restrict the ambient manifold to dimension 4. However, the dimension restriction is essential to the following theorem, where the $\mu-$bubble technique is needed to get a diameter bound using positive scalar curvature.

Theorem 2.2.8. *Consider* $(X^4, \partial X)$ *a complete manifold with scalar curvature* $R \geq 2$*, and* $(M, \partial M) \hookrightarrow (X, \partial X)$ *a two-sided stable immersed free boundary minimal hypersurface. Let* N *be a component of* $\overline{M \setminus K}$ *for some compact set* K*, with* $\partial N = \partial_0 N \cup \partial_1 N, \partial_0 N \subset \partial M$ *and* $\partial_1 N \subset K$*. If there is* $p \in N$ *with* $d_N(p, \partial_1 N) > 10\pi$*, then we can find a Caccioppoli set* $\Omega \subset B_{10\pi}(\partial_1 N)$ *whose reduced boundary has that: any component* Σ *of* $\overline{\partial\Omega \setminus \partial N}$ *has diameter at most* 2π *and intersect with* $\partial_0 N$ *orthogonally.*

2.3 Curvature Estimates and Weakly Bounded Geometry

We start with the first consequence, curvature estimates for free boundary stable minimal hypersurface in manifolds with weakly bounded geometry.

Lemma 2.3.1. *Let* $(X^n, \partial X, g)$ *be a complete Riemannian manifold with weakly bounded geometry, and* $(M^{n-1}, \partial M) \hookrightarrow (X, \partial X)$ *a complete stable immersed free boundary minimal hypersurface, then*

$$\sup_{q \in M} |\mathrm{I\!I}(q)| \leq C < \infty,$$

for a constant $C = C(X, g)$ *independent of* M*.*

Proof. We follow the proof as given in [15]. We prove that for any compact set $K \subset M$, we have the following curvature estimates:

$$\max_{q \in K} |\mathrm{I\!I}(q)| \min\{1, d_M(q, \partial_1 K)\} \leq C < \infty, \tag{2.3.1}$$

with $\partial M \cap K = \partial_0 K$ and $\partial K \setminus \partial M = \partial_1 K$,

Towards a contradiction, assume there is a sequence of compact sets $K_i \subset M_i \hookrightarrow X$ the latter being a complete stable immersed free boundary minimal hypersurface, and

$$\max_{q \in K_i} |\mathrm{I\!I}_i(q)| \min\{1, d_{M_i}(q, \partial_1 K_i)\} \to \infty. \tag{2.3.2}$$

Then by compactness of K_i we can find $p_i \in K_i \setminus \partial_1 K_i$ with

$$|\mathbb{I}_i(p_i)| \min\{1, d_{M_i}(p_i, \partial_1 K_i)\} = \max_{q \in K_i} |\mathbb{I}_i(q)| \min\{1, d_{M_i}(q, \partial_1 K_i)\} \to \infty. \tag{2.3.3}$$

Define $r_i := |\mathbb{I}_i(p_i)|^{-1} \to 0$ and x_i the image of p_i in X. Using the weakly bounded geometry assumption and a pullback operation as in [15] Appendix B, we can find a sequence of pointed 3-manifolds (S_i, s_i), local diffeomorphisms $\Psi_i : (S_i, s_i) \to (K_i, p_i)$ with the boundary components mapped correspondingly $\Psi_i(\partial_l S_i) = \partial_l K_i (l = 0, 1)$, and immersions $F_i : (S_i, s_i) \hookrightarrow (B(a_i, Q^{-1}) \cap \mathbb{H}_+, a_i)$ so that the following diagram commutes (writing $B_i := B(a_i, Q^{-1}) \cap \mathbb{H}_+$), and that $F_i : S_i \to (B_i, \Phi_i^\star g)$ is a two-sided stable minimal immersion, in the free boundary sense along $\partial_0 S_i$ but not $\partial_1 S_i$,

$$\begin{array}{ccc} S_i & \xrightarrow{F_i} & B_i \\ {\scriptstyle \Psi_i}\downarrow & & \downarrow{\scriptstyle \Phi_i} \\ M_i & \longrightarrow & X. \end{array}$$

Note that in the weakly bounded geometry condition we may also require the Euclidean norm of a_i is no more than Q^{-1}.

We can now consider the blow-up sequence

$$\tilde{F}_i : (S_i, s_i) \to (\hat{B}_i, a_i), \;\; \hat{B}_i = B(a_i, r_i^{-1} Q^{-1}) \cap \mathbb{H}_+ \text{ with metric } r_i^{-2} \Phi_i^\star g. \tag{2.3.4}$$

By assumption of weakly bounded geometry, $(\hat{B}_i, a_i)$ converges to the Euclidean metric in $C^{1,\alpha}$ on any compact sets. We now consider S_i with metric induced from $\tilde{F}_i$. By the point picking argument, for any point q in a ball of fixed radius $R > 0$ around s_i, we have a uniform bound on $|\tilde{\mathbb{I}}_{S_i}(q)| \leq C(R)$. The weakly bounded geometry condition then gives $|\hat{\mathbb{I}}_{S_i}(q)| \leq C'(R)$ for the immersion $\hat{F} : (S_i, s_i) \to (\hat{B}_i, a_i)$, the latter with Euclidean metric g_0. This allows us to write a connected component of $B_\mu^{S_i}(q)$ as a graph of a function f_i over a subset $B_r(0) \cap \mathbb{H}_i$ of $T_q S_i$ for some $\mu, r > 0$, here $B_r(0)$ is the Euclidean ball and $\mathbb{H}_i$ is some halfspace in $\mathbb{R}^3$ that may not go through the origin.

Now following the same argument as in [15], we know that the functions f_i have uniformly bounded $C^{2,\alpha}$ norm. To continue the argument as in [15], we can extend the graph f_i from $B_r(0) \cap \mathbb{H}_i$ to all of $B_r(0)$ and f_i still has uniformly bounded $C^{2,\alpha}$ norm (but the extended part is not minimal as a hypersurface in $\hat{B}_i$). This gives us that on any bounded set, (S_i, s_i) has injectivity radius bounded away from 0 and bounded sectional curvature, with respect to the metric $(\tilde{F}_i)^{\star}(r_i^{-2}\Phi_i^{\star} g)$.

Then we can use the same argument in [15] and pass to the limit, to get a subsequence converging to a complete minimal immersion (S_∞, s_∞) in $\mathbb{R}^4$, or one that is minimal on $\mathbb{H}_+$ and that intersect the $\partial\mathbb{H}_+$ orthogonally, furthermore $|\mathrm{I\!I}_\infty(s_\infty)| = 1$ (note that under this blow-up sequence, $\tilde{\mathrm{I\!I}}_i(s_i) = 1$ by the choice of r_i and $\tilde{d}_{S_i}(s_i, \partial_1 S_i) \to \infty$). In the latter case we can use reflection principle (see for example Guang-Li-Zhou [32]) and reduce to a complete minimal immersion in $\mathbb{R}^4$, which is a contradiction to the result of Chodosh and Li ([14],[12])– any complete two-sided stable minimal hypersurface in $\mathbb{R}^4$ is flat. □

Remark 2.3.2. *The pullback operation in [15] applies to open manifolds without boundary(an interior ball of small radius in K_i near p_i), in our case for the proof above, we need to extend over the free boundary part of this small ball, apply [15] to the extended open manifold and one can check that we still get a free boundary immersion near $\partial_0 S_i$.*

Now we prove the following volume control theorem for a manifold with weakly bounded geometry. This argument follows as in Lemma 2.1 in Bamler-Zhang [4]. In this paper given an immersion $M \hookrightarrow X$, we write the intrinsic distance function as $d_M(\cdot,\cdot)$ and extrinsic distance function as $d_X(\cdot,\cdot)$.

Lemma 2.3.3. *Let $(X^n, \partial X, g)$ be a complete Riemannian manifold with weakly bounded geometry at scale Q, and $(M^{n-1}, \partial M) \hookrightarrow (X, \partial X)$ a complete immersed submanifold with bounded second fundamental form, then the following is true,*

- *there is $0 < N < \infty$ such that for any $p \in M$, the maximum number of disjoint balls of radius δ centered around points in $B^M_{4\delta}(p)$ is bounded by N,*
- *for any $R > 0$, there is a constant $C = C(R, Q)$ such that the volume of balls of radius R around any point in M is bounded by C.*

Proof. To prove the first claim, we first prove that there is a fixed $0 < r_0 < Q^{-1}$ such that for any point p in M, we have for any $r < r_0$, $\Psi(B_r^S(s)) = B_r^M(p)$, here Ψ comes from applying the pullback operation as in the previous lemma, i.e. we have the following commutative diagram, with local diffeomorphism $\Psi : (S, s) \to (M, p)$ and immersion $F : (S, s) \to (B, a)$, with $B = B_{Q^{-1}}(a) \cap \mathbb{H}_+$,

$$\begin{array}{ccc} (S,s) & \xrightarrow{F} & (B,a) \\ \Psi \downarrow & & \downarrow \Phi \\ (M,p) & \longrightarrow & (X,x) \end{array}$$

Note since image of any path in $B_r^S(s)$ is again a path in $B_r^M(p)$ and Ψ is a local isometry, we have $\Psi(B_r^S(s)) \subset B_r^M(p)$. To prove the other direction, we look at a point q connected to p by a shortest path of unit speed $I(t) : [0, l] \to M (l < r)$, again since Ψ is a local isometry we can find a path in S with unit speed $J(t) : [0, \epsilon] \to S, J(0) = s$, that is mapped isometrically to I under Ψ. Writing $\mathrm{Im}(I)$ for the image of $I(t)$ in M, we note that the preimage $\Psi^{-1}(\mathrm{Im}(I))$ is a union of paths in S since Ψ is a local isometry, one of the component must contain $J(t)$, which we denote as $J(t)$ from now on. The length of J (denoted as t_0) is at least l, since if not, then as $t \to t_0$, $\Psi(J(t))$ converges to a point on the path $I(t)$, whose preimage in J still lies in $B_r^S(s)$ and can be used to extend J longer. Therefore J must also reach a preimage of q at length $l < r$. So we get $B_r^M(p) \subset \Psi(B_r^S(s))$.

We now prove the first claim. Let $8\delta < r_0$, then we have that $\Psi(B_{4\delta}^S(s)) = B_{4\delta}^M(p)$ by the above proof. For any disjoint balls $B_\delta^M(p_i)$ with $p_i \in B_{4\delta}^M(p)$, we must have $s_i \in B_{4\delta}^S(s)$, so that $\Psi(B_\delta^S)(s_i) = B_\delta^M(p_i)$, therefore $B_\delta^S(s_i)$ are disjoint.

Note that $S \to B$ also has bounded second fundamental form, and the weakly bounded geometry assumption says the pullback metric via Φ is comparable to the Euclidean metric as two forms, which implies that the volume of $B_{5\delta}^S(s)$ is bounded above by $C\delta^{n-1}$, and the volume of $B_\delta^S(s_i)$ is bounded from below by $C'\delta^{n-1}$ for some constant C, C' depending on Q(here we may choose r_0 to be even smaller depending on the second fundamental form). Therefore, the number of such points s_i is bounded by a fixed constant N, and so is the number of p_i.

We now prove the second claim. We want to bound the volume of $B_R^M(p)$ for any given $R > 0$ and any $p \in M$, and we may assume $R > r_0 > 8\delta$. Let $(B_\delta^M(p_i))_{i=1}^k$ be a choice of pairwise disjoint balls with centers in $B_{4\delta}^M(p)$ and with the maximum k ($k \le N$). By maximality,

$$B_{4\delta}^M(p) \subset \cup_{i=1}^k B_{2\delta}^M(p_i). \tag{2.3.5}$$

We now argue that for all $r \ge 4\delta$,

$$B_{2\delta+r}^M(p) \subset \cup_{i=1}^k B_r^M(p_i). \tag{2.3.6}$$

Consider a point $y \in B_{2\delta+r}^M(p)$, and a path $\gamma(t)$ (reparametrized by arc length) from p to y with length $l < r + 2\delta$. Then by (2.3.5) there is some point p_i so that $\gamma(4\delta) \in \overline{B_{2\delta}^M(p_i)}$. We have,

$$d(p_i, y) \le l - 4\delta + d(\gamma(4\delta), p_i) \le l - 2\delta < r,$$

completing the proof of (2.3.6).

We now prove by induction that for any $k \ge 2$ and any $q \in M$, the volume of $B_{2\delta k}^M(q)$ is bounded by a constant C^k with $C = C(Q, N, \delta)$. For $k = 2$, this is already proved in the first claim. Now assuming the claim is true for some $k \ge 2$, then using (2.3.6) for $r = 2\delta k$ gives,

$$|B_{2(k+1)\delta}^M(q)| \le NC^k \le C^{k+1}.$$

Choosing k large enough we can bound the volume of $B_R^M(q)$ for any given $R > 0$. $\square$

2.4 Parabolicity on Manifolds with Boundary

Given a manifold with boundary $(M^n, \partial M)$, and any continuous submanifold E^n, recall we reserve the notation ∂E to denote the manifold boundary of E (instead of as a subset in M). Therefore we can decompose $\partial E = \partial_1 E \cup \partial_0 E$ where $\partial_0 E = \partial E \cap \partial M$ and $\partial_1 E = \overline{\partial E \setminus \partial M}$. And we say that $\partial_1 E \cap \partial_0 E$ at an angle $\theta(x) \in (0, \pi)$, if for any $x \in \partial_1 E \cap \partial_0 E$ the hyperplane $T_x\partial_1 E$ and $T_x\partial_0 E$ intersect at angle $\theta(x)$ in the interior of E. In this paper we only consider domains E in M that are smooth except at the intersections $\partial_1 E \cap \partial_0 E$, we call these points corner points.

Definition 2.4.1. *Consider $(M^n, \partial M)$ complete manifold with noncompact boundary. An end of $(M, \partial M)$ is a sequence of complete continuous n-dimensional submanifold $(E_k)_{k\geq 0}$ with boundary, where each E_k is a noncompact connected component of $M \backslash C_k$ for compact continuous submanifold $C_k \subset C_{k+1}$, and $E_{k+1} \subset E_k$.*

When $C_k = C_{k+1} = K$ and $E_k = E_{k+1} = E$ for all $k \geq 0$, we will also call E an end with respect to the compact set K.

Definition 2.4.2. *For any end E of M, we say that ∂E intersect with the boundary ∂M transversally (or at an angle $\theta(x) \in (0, \pi)$) if $\partial_0 E$ and $\partial_1 E$ intersect transversally (or at an angle $\theta(x)$) as submanifolds in M, that is, for any point $x \in \partial_1 E \cap \partial_0 E$, the tangent planes $T_x\partial_0 E$ and $T_x\partial_1 E$ are not equal (or at an angle $\theta(x)$).*

In the following theorem we show how we can purturb the angle of intersection of an end in an arbitrarily small neighborhood.

Theorem 2.4.3. *Consider $(M^n, \partial M)$ complete orientable manifold with noncompact boundary and let $d_M(p, \cdot)$ be the continuous distance function from a fixed point $p \in M$ (we will mollify it to be smooth on a compact set in M without changing the notation). Then for almost every $c > 0$, the preimage $E_c = d_M^{-1}([c, \infty))$ is a submanifold with boundary and intersects with the boundary ∂M transversally. Furthermore given any $\delta > 0$ and constant $\theta \in (0, \frac{\pi}{2})$, we can find another continuous submanifold E within the δ-neighborhood of E_c so that the angle between the tangent planes $T_x\partial_1 E$ and $T_x\partial_0 E$ is equal to θ. The submanifold E is smooth except at the corners.*

Proof. We first consider the continuous distance function $h = d_M(p, \cdot)$, for any $N > 0$ and any $\delta > 0$, there is a mollification $\bar{h}$ such that $\bar{h}$ is smooth on $B_N^M(p)$ and $\|h - \bar{h}\|_{L^\infty(B_N^M(p))} < \delta/2$. Then it is a standard proof (see for example in [33] section 2.1) that for almost every $0 < c < N$, the map $d\bar{h}_x : T_xM \to \mathbb{R}$ and the map $d\bar{h}_x : T_x\partial M \to \mathbb{R}$ are both nonzero, and the preimage $E_c = \bar{h}^{-1}([c, \infty)$ is a continuous submanifold intersecting ∂M transversally and is smooth except at the corners. We now show that we can purturb to arrange the angle of interesection to be any constant $\theta \in (0, \frac{\pi}{2})$ in a $\frac{\delta}{2}-$neighborhood of E_c.

We denote the intersection $\partial M \cap \partial E_c =: I$, note I is orientable because it's the preimage of the regular value s of the function $d(p, \cdot)$ restricted to the boundary by [38] Proposition 15.23. Using a unit normal vector field μ of $I \subset \partial M$ that is outwarding pointing with respect to E_c, we find a local coordinates (z, t) within the $\delta'-$neighborhood of $I \subset \partial M(\delta'$ to be decided), here (z, t) means $(z, 0) \in I$ and (z, t) stands for the point $\exp^{\partial M}_{(z,0)}(t\mu)$(the exponential map on ∂M). Now similarly using the outward pointing unit normal ν of $\partial M \subset M$, we build a local coordinates denoted as $(z, t, r) = \exp^M_{(z,t)}(r\nu)$. Denote the projection map onto the last coordinate r as $P_r : E_c \to \mathbb{R}$, then 0 is a regular value of P_r because if $dP_r(x) : T_xE_s \to \mathbb{R}$ is zero for some point $x \in I = P_r^{-1}(0)$, then $T_xE_c \subset T_x\partial M$, contradiction to the transversal intersection of them we just proved. Further note dP_r is zero restricted to $T_x\partial M$, espeically in the directions on T_xI. Now fix a point $(z_0, 0, 0) \in I$, and consider the slice $S_{z_0} = \{(z, t, r) \in E_c, z = z_0\}$ in the $rt-$plane, then P_r restricted to S_{z_0} has that dP_r is nonzero around a neighborhood of origin, so the tangent line along S_{z_0} is never parallel to the t-axis in this neighborhood, meaning we can write S_{z_0} as a graph $(z_0, t(r), r)$ in this neighborhood (the function $t(r) = t_{z_0}(r)$ also depends on z_0 but we omit the notation).

Now we can concatenate the graph $t(r)$ with the linear map $\bar{t}(r) = \tan(\theta)r$, at $r = \delta''$ for some $\delta'' < \delta'$, to get a new function $\hat{t}(r)$ with jump singularity at $r = \delta''$, and using a bump function $\phi(r)$ supported near the singularity, we have the function $\hat{t}(r)(1-\phi(r))$ gives the graph bounding our desired E together with E_c. Given a fixed $\theta \in (0, \frac{\pi}{2})$, we can choose δ', δ'' small enough so that the modification happens within the $\frac{\delta}{2}-$neighborhood of E_c. □

From now on, in this section we will mostly follow the discussion in [15] where the case is for manifolds without boundary.

Definition 2.4.4 (Parabolic Component)**.** *Let $(M^n, \partial M)$ be a complete Riemannian manifold with noncompact boundary, and E an end with respect to some compact K. We say that E is parabolic if there is no positive harmonic function $f \in C^{2,\alpha}(E)$, for* *some $\alpha > 0$, so that,*

$$f\big|_{\partial_1 E} = 1, \quad \partial_\nu f\big|_{\partial_0 E} = 0, \quad f\big|_{E^\circ} < 1,$$

with ν the outward pointing unit normal of ∂M.

Otherwise we say that E is nonparabolic.

We note that if E is nonparabolic, then there is a harmonic function f on E that is $C^{2,\alpha}$ across the corners, in the sense that it can be extended to an open neighborhood of E in M.

We first deal with the regularity issue arising in the above definition. That is, when $\partial_1 E \cap \partial_0 E \neq \emptyset$, a weakly harmonic function over E may not lie in the class $C^2(E)$ or even $C^1(E)$. The following theorem says that if we purturb the angle of intersection of $\partial_1 E \cap \partial_0 E$ to be small, we will have enough regularity.

Theorem 2.4.5. *Consider a connected compact Riemannian manifold with boundary $(K, \partial K = \partial_1 K \cup \partial_0 K)$, and $\partial_1 K$ intersect with $\partial_0 K$ transversally as smooth codimension 1 submanifolds, with constant angle $\theta \in (0, \pi/4)$ contained in K. We write ν as the outward pointing unit normal at each boundary (ν exists almost everywhere, i.e. except at the corner points). Then a weakly harmonic function $f \in W^{1,2}(K)$ with prescribed boundary condition: $f|_{\partial_1 K} = g|_{\partial_1 K}$, and $\nabla_\nu f|_{\partial_0 K} = \nabla_\nu g|_{\partial_0 K}$ with $g \in C^{2,\alpha(\theta)}(K)$, is also $C^{2,\alpha(\theta)}$ for some fixed $\alpha(\theta) > 0$.*

Proof. The function $u = g - f$ satisfies $\Delta u = \Delta g =: h$ and has Dirichlet boundary condition over $\partial_1 K$ and Neumann boundary condition over $\partial_0 K$. Then u is the unique solution to the following problem, in a complete subspace of $W^{1,2}(K)$, namely

$$\int_K \nabla u \cdot \nabla \phi = -\int_K h\phi, \quad \forall \phi \in C_c^\infty(K \setminus \partial_1 K),$$

over the set $\mathcal{S} := \{u \in W^{1,2}(K), u|_{\partial_1 K} = 0\}$.

We note that a unique solution exists by Lax-Milgram, and we have that the $W^{1,2}$ norm of the solution u is finite since,

$$\int_K \nabla u \cdot \nabla u = -\int_K hu \leq \|h\|_{L^2}\|u\|_{L^2} \leq C\|h\|_{L^2}\|\nabla u\|_{L^2},$$

where in the last step we used Poincaré inequality since $u|_{\partial_1 K} = 0$ ($\partial_1 K \neq \emptyset$). So away from the corners we can continue with standard iteration scheme (see for example [19], [1] and [23]) to get for any $k \in \mathbb{N}$, $\|u\|_{H^k} \leq C'\|u\|_{H^1} \leq C(h, K)$, where $\|u\|_{H^k} := \|\nabla^k u\|_{L^2(K)}$. We briefly write the process using partition of unity here.

Given any interior ball $B_r \subset B_R \subset K^\circ$ consider a bump function supported on B_R and $\phi = 1$ on B_r. Then $\Delta(\phi u) = (\Delta\phi)u + 2\nabla u \cdot \nabla\phi + h\phi \in L^2$, so we have that $\|\phi u\|_{H^2} \leq C'(\|\Delta(\phi u)\|_{L^2} + \|u\|_{H^1}) \leq C(R, r, h)\|u\|_{H^1}$. Differentiating the equation again and iterate the process, we get the claimed bounds on H^k norm of u on B_r. So we can get C^∞_{loc} bounds on any compact set in the interior.

A similar process holds if $B_r \subset B_R$ are balls centered around a boundary point $B_R \cap \partial_0 K = \emptyset$. Consider ϕf with ϕ compactly supported in B_R but is equal to 1 on B_r (including points on the boundary), look at ϕu on B_R (and flatten the intersetion of B_R and $\partial_1 K$, this is not an issue since we only want to bound u in B_r). Then the same process as above applies using boundary estimates.

For purely Neumann condition a similar treatment holds. We need to choose bump functions ϕ supported in boundary coordinates charts, so that on the boundary of B_R, $\phi = 1, \partial_\nu \phi = 0$, to make sure $\partial_\nu(\phi f) = 0$ on the boundary of B_R (again we flatten the intersection of B_R and $\partial_0 K$). Then using boundary estimates for Neumann conditions, we again have the above property.

If B_R is a ball centered around a point on the corners: $\partial_1 K \cap \partial_0 K$, we have $\Delta u = h$ on B_R, using normal coordinates for small r, the function u solves a uniformly elliptic nonhomogeneous equation, both in the weak sense and classically everywhere except at the corners. We choose a smooth bump function ϕ like in the Neumann case, i.e. $\phi = 1$ and $\partial_\nu \phi = 0$ on the boundary of B_R. Then by the work of Miranda [47], $u\phi$ is (Hölder) continuous (and bounded) on B_R, and under this assumption, using the method of barrier functions, Azzam [2] gives that $u \in C^{2,\alpha(\theta)}(B_r)$ for $\theta \in (0, \pi/4)$. The following bounds holds on B_r for $r < \frac{R}{2}$ ([2]):

$$\sup_{x \in B_r} |u(x)| + \sup_{x,y \in B_r} \frac{|D^2 u(x) - D^2 u(y)|}{d_K(x,y)^\alpha} \leq C \tag{2.4.1}$$

where the constant only depend on the manifold K, the function g and the constant

α. In particular, on any compact set in K, u has bounded $C^{2,\alpha}$ norm and so does f, i.e. $\|f\|_{C^{2,\alpha}}(K) \leq C(g, K)$. We will make use of the bound soon. $\square$

Remark 2.4.6. *The book of Miranda [48], the paper of Liebermann [44] and of Azzam and Kreyszig [3] give a nice review over regularity of solutions of mixed boundary value problem.*

In this paper, when we say that an end is parabolic or non-parabolic, we always mean that $\partial_1 E \cap \partial_0 E$ with a constant angle in $(0, \pi/4)$. Applying Hopf Lemma (see [23] Lemma 3.4) we have the following maximum principle.

Theorem 2.4.7. *If K is compact in M, and f is harmonic on K with $\partial_\nu f|_{\partial_0 K} = 0$, then*

$$\max_{\partial_0 K} f \leq \max_{\partial_1 K} f, \quad \min_{\partial_0 K} f \geq \min_{\partial_1 K} f.$$

In particular, $\max_K f = \max_{\partial_1 K} f$ and $\min_K f = \min_{\partial_1 K} f$.

Lemma 2.4.8. *Let $(M, \partial M)$ be a complete Riemannian manifold. Let $K \subset M$ be a compact subsest of M. Let $E \subset M$ be an unbounded component of $M \setminus K$, fix $p \in E$ and consider $B_{R_i}(p)$. Assume E is parabolic, then there are positive harmonic functions f_i on $E \cap B_{R_i}$ with*

$$f_i|_{\partial_1 E} = 1, \nabla_\nu f_i|_{\partial_0 E} = 0, f_i|_{\partial_1 B_{R_i}} = 0,$$

with $R_i \to \infty$. Then $f_i \to 1$ in $C^{2,\alpha}_{\text{loc}}(E)$ and $\lim_i \int_E |\nabla f_i|^2 = 0$.

Remark 2.4.9. *Again we may choose R_i and mollify the boundary $\partial B_{R_i} \cap \partial M$ without relabeling so that the angle of intersection is $\theta \in (0, \pi/4)$. We will omit this step later when mollification is needed.*

Proof. Let f_i be the minimizer of Dirichlet energy over B_{R_i} given the above boundary conditions. We first claim that f_i has finite and decreasing Dirichlet energy. Since given a Lipschitz domain in $\mathbb{R}^n$, a function is in $W_0^{1,2}$ (zero trace) if and only if it can be approximated by a sequence of compactly supported smooth functions, and E has Lipschitz boundary, using a partition of unity, the same holds for on E. So if we extend f_1 by zero on $B_{R_i} \setminus B_{R_1}$ we get another candidate and that we may assume $\int_E |\nabla f_{i+1}|^2 \leq \int_E |\nabla f_i|^2 \leq \int_E |\nabla f_1|^2 = C_1$.

Using Lemma 2.4.5 and maximum principle, we know that $\|f_i\|_{C^0} \leq 1$, for all $i \geq 0$. Now using equation (2.4.1), we know that $\sup_i \|f_i\|_{C^{2,\alpha}(K')}$ is finite for any compact subset $K' \subset E$.

We also have that f_i subsequentially converge in $C^{2,\alpha}_{loc}$ (for some $\alpha > 0$) to a harmonic function $1 \leq f \leq 0$ on E, and by parabolicity and maximum principle, $f = 1$ everywhere on E, and we have:

$$\int_E |\nabla f_i|^2 = \int_{\partial_1 E} f_i \nabla_\nu f_i \to 0,$$

using the uniform convergence to $f = 1$ in C^1_{loc}-norm near $\partial_1 E$. □

We note that nonparabolicity is inherited by subsets. The proof of the lemma below is analogous to Proposition 3.5 in [15] if we use Lemma 2.4.5 to deal with regularity of mixed boundary value problem.

Lemma 2.4.10. *Consider $K \subset \hat{K}$ compact subset in $(M, \partial M)$, with each component of $M \setminus \hat{K}$ and $M \setminus K$ is smooth except at the corners. If E is a nonparabolic component of $M \setminus K$, then there is a nonparabolic component of $M \setminus \hat{K}$.*

The above lemma, together with Theorem 2.4.3 says that, starting with any nonparabolic end $E_1 := E \subset M \setminus K$, we can build a sequence of nonparabolic sets E_k with $\partial_1 E_k \cap \partial M$ contained correspondingly in any small neighborhood of $\partial B_{R_k}(p)$, intersecting with ∂M at angle θ for any $\theta \in (0, \pi/4)$, for any R_k in a open dense set of $(0, \infty)$. Hence we have the following definition.

Definition 2.4.11 (Nonparabolic Ends)**.** *Let (E_k) be an end with each ∂E_k intersecting with ∂M at angle $\theta \in (0, \pi/4)$ and smooth except at the corners, we say that (E_k) is a nonparabolic end if $k \geq 0$, the component E_k is nonparabolic.*

We also note that the unique minimal barrier function on a nonparabolic end has finite Dirichelt energy, a fact we will use in Section 2.5.

Theorem 2.4.12. *If E is a nonparabolic end of M, then there is a harmonic function f over E with $f|_{\partial_1 E} = 1$ and $\nabla_\nu f|_{\partial_0 E} = 0$, that is minimal among all such harmonic functions and has finite Dirichlet energy.*

Proof. By definition of nonparabolicity, there is a positive harmonic function g with $g|_{\partial_1 E} = 1, \partial_\nu g|_{\partial_0 E} = 0$. We solve over an exhaustion $\cup_{i\in\mathbb{N}}\Omega_i = E$, the following mixed boundary value problem (each Ω_i contains $\partial_1 E$),

$$\Delta f_i = 0, \quad f_i|_{\partial_1 E} = 1, \quad \partial_\nu f_i|_{\partial_0 E} = 0, \quad f_i|_{\partial\Omega_i\setminus(\partial_1 E\cup\partial_0 E)} = 0.$$

We may assume all the corners of Ω_i has interior angle in $(0, \pi/4)$. Maximum principle then gives that $f_i \leq g$ over Ω_i. Using the same argument as in Lemma 2.4.8, we have that f_i converge in $C^{2,\alpha}_{\text{loc}}$ to a positive barrier function over E, that is bounded by g. Since this argument applies for arbitrary g, we have that f is the unique minimal barrier function. Now we show f has finite Dirichlet energy.

$$\int_{\Omega_i} |\nabla f_i|^2 = \int_{\partial_1 E} f_i \nabla_\nu f_i \leq C_0,$$

where the last inequality is bounded by a constant we again used equation (2.4.1) near a compact set containing $\partial_1 E$. Now we can let $i \to \infty$ in the equation below to get that f has finite Dirichlet energy.

$$\int_{\Omega_i} |\nabla f|^2 = \lim_{l>i} \int_{\Omega_i} |\nabla f_l|^2 \leq C_0.$$

□

2.5 At Most One Nonparabolic End

We follow the same method in [15] to show that under a suitable condition ($A_2 \geq 0$) for the boundary ∂X of an ambient manifold X with $\text{Ric}_2 \geq 0$, any free boundary stable minimal hypersurface with infinite volume can only have at most 1 nonparabolic end. We begin with the following theorem.

Theorem 2.5.1. *Consider $(M, \partial M)$ a complete manifold, $K \subset M$ compact and E_1, E_2 are two nonparabolic components of $M \setminus K$. Then there is a nonconstant bounded harmonic function with finite Dirichlet energy on M.*

Proof. By definition of parabolicity, on each end $E_s (s = 1, 2)$ we can find a harmonic function $1 \geq h_s(x) > 0$ with $h_s|_{\partial_1 E} = 1, \partial_\nu h_s|_{\partial_0 E} = 0$. Using Lemma 2.4.12, we may assume that each h_s has finite Dirichlet energy.

We solve for harmonic functions f_i on B_{R_i} (again mollifying the boundary to get small intersection angle with ∂M) such that $f_{\partial_1 B_{R_i} \cap E_1} = h_1$, $f_{\partial_1 B_{R_i} \cap E_2} = 1 - h_2$, $f_i = 0$ on other components of $\partial_1 B_{R_i}$, and $\partial_\nu f_i|_{\partial_0 M} = 0$. Using a similar argument to that in section 4, we have that

$$\sup_i \|\nabla f_i\|^2_{L^2(B_{R_i})} \leq C(\|\nabla f_1\|^2_{L^2(B_{R_1})} + \|\nabla h_1\|^2_{L^2} + \|\nabla h_2\|^2_{L^2}) < \infty,$$

and that f_i converges in $C^{2,\alpha}_{\text{loc}}$ to a harmonic function on M with finite Dirichlet energy. The function takes value in $[0, 1]$ by maximum principle, and is nonconstant by arrangement at the two ends E_1, E_2. □

Theorem 2.5.2. *Let $(X^4, \partial X)$ be a complete manifold with $\mathrm{Ric}_2 \geq 0$, and $(M^3, \partial M)$ a free boundary orientable stable minimal immersion, given a smooth harmonic function u on M with Neumann boundary condition, we have the following estimates:*

$$\begin{aligned} &\frac{1}{3}\int_M \phi^2 |\mathrm{I\!I}|^2 |\nabla u|^2 + \frac{1}{2}\int_M \phi^2 |\nabla|\nabla u||^2 \\ &\leq \int_M |\nabla \phi|^2 |\nabla u|^2 + \int_{\partial M} |\nabla u| \nabla_\nu |\nabla u| \phi^2 - A(\eta, \eta) |\nabla u|^2 \phi^2. \end{aligned}$$

Here $\mathrm{I\!I}$ is the second fundamental form of $M \to X$ and A is the second fundamental form of $\partial X \to X$, $\nu \perp T\partial M$ in TM and $\eta \perp M$ in X.

If we have $A_2 \geq 0$, then:

$$\frac{1}{3}\int_M \phi^2 |\mathrm{I\!I}|^2 |\nabla u|^2 + \frac{1}{2}\int_M \phi^2 |\nabla|\nabla u||^2 \leq \int_M |\nabla \phi|^2 |\nabla u|^2 \tag{2.5.1}$$

Proof. Using the second variation for orientable hypersurfaces we have for any family of immersion with speed $\frac{d}{dt}\big|_{t=0}\varphi_t(M) = \phi\eta$:

$$\begin{aligned}0 &\leq \frac{d^2}{dt^2}\Big|_{t=0}\text{Area}(\varphi_t(M)) \\ &= \int_M |\nabla_M \phi|^2 - (|\mathrm{I\!I}|^2 + \text{Ric}(\eta,\eta))\phi^2 - \int_{\partial M} A(\eta,\eta)\phi^2\end{aligned}$$

Fixing any compact supported smooth function ϕ, we plug in $\sqrt{|\nabla u|^2 + \epsilon}\phi$ to the second variation formula then let $\epsilon \to 0$ to get the following,

$$\begin{aligned}0 \leq &\int_M |\nabla\phi|^2|\nabla u|^2 + \phi^2|\nabla|\nabla u||^2 + \langle\nabla\phi^2, \nabla|\nabla u|\rangle|\nabla u| - |\mathrm{I\!I}|^2|\nabla u|^2\phi^2 \\ &- \int_{\partial M} |\nabla u|^2 A(\eta,\eta)\phi^2 \\ &= \int_M |\nabla\phi|^2|\nabla u|^2 - |\nabla u|\Delta|\nabla u|\phi^2 - |\mathrm{I\!I}|^2|\nabla u|^2\phi^2 \\ &+ \int_{\partial M} \phi^2(-|\nabla u|^2 A(\eta,\eta) + |\nabla u|\nabla_\nu|\nabla u|),\end{aligned}$$

here we have used that $\text{Ric}_2 \geq 0$ implies $\text{Ric}_X \geq 0$. Note over M° we have the following (see also [15]):

$$\begin{aligned}\Delta|\nabla u|^2 &= 2\,\text{Ric}(\nabla u, \nabla u) + 2|\nabla^2 u|^2, &&\text{Bochner's Formula} \\ |\nabla^2 u|^2 &\geq \frac{3}{8}|\nabla u|^{-2}|\nabla|\nabla u|^2|^2, &&\text{Improved Kato's Inequality} \\ \text{Ric}(\nabla u, \nabla u) &\geq \frac{-2}{3}|\mathrm{I\!I}|^2|\nabla u|^2, &&\text{Lemma 2.2.5}\end{aligned}$$

These together imply $|\nabla u|\Delta|\nabla u| \geq \frac{-2}{3}|\mathrm{I\!I}|^2|\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2$, which we can plug into the last inequality, to get:

$$\int_M \frac{1}{3}|\mathrm{I\!I}|^2|\nabla u|^2\phi^2 + \frac{1}{2}|\nabla|\nabla u||^2\phi^2 \leq \int_M |\nabla\phi|^2|\nabla u|^2 + \int_{\partial M} \phi^2(|\nabla u|\nabla_\nu|\nabla u| - |\nabla u|^2 A(\eta,\eta)).$$

Note using Neumann condition we get $0 = \nabla_{\nabla u}\langle\nabla u, \nu\rangle = \langle\nabla_{\nabla u}\nabla u, \nu\rangle + \langle\nabla u, \nabla_{\nabla u}\nu\rangle$.

So we can compute the boundary terms:

$$\begin{aligned}
&\int_{\partial M} |\nabla u|\nabla_\nu|\nabla u|\phi^2 - A(\eta,\eta)|\nabla u|^2\phi^2 \\
=&\int_{\partial M} -|\nabla u|^2(\langle\frac{\nabla u}{|\nabla u|},\nabla_{\frac{\nabla u}{|\nabla u|}}\nu\rangle + A(\eta,\eta))\phi^2) \\
=&\int_{\partial M} -|\nabla u|^2(\langle\frac{\nabla u}{|\nabla u|},\nabla_{\frac{\nabla u}{|\nabla u|}}\nu\rangle + \langle\eta,\nabla_\eta\nu\rangle)\phi^2
\end{aligned}$$

Using that $A(e_1,e_1)+A(e_2,e_2)\geq 0$ if $e_1\perp e_2$ (note $\eta\perp M$ while ∇u is along M), the above integrand over the boundary is now nonnegative, and we have the inequality:

$$\frac{1}{3}\int_M \phi^2|\mathrm{I\!I}|^2|\nabla u|^2 + \frac{1}{2}\int_M \phi^2|\nabla|\nabla u||^2 \leq \int_M |\nabla\phi|^2|\nabla u|^2$$

□

Theorem 2.5.3. *Let $(X^4,\partial X)$ be a complete manifold with $\mathrm{Ric}_2\geq 0$, and the boundary of X has second fundamental form satisfying $A_2\geq 0$, and $(M,\partial M)$ a free boundary orientable stable minimal immersion with infinite volume, then for any compact set $K\subset M$, there is at most 1 nonparabolic component in $M\setminus K$. In particular, M has at most one non-parabolic end.*

Proof. Since we can apply inequality (2.5.1) of Theorem (2.5.2), we have, for any compactly supported smooth function ϕ,

$$\frac{1}{3}\int_M \phi^2|\mathrm{I\!I}|^2|\nabla u|^2 + \frac{1}{2}\int_M \phi^2|\nabla|\nabla u||^2 \leq \int_M |\nabla\phi|^2|\nabla u|^2.$$

We can proceed as in [15]. Suppose there are two nonparabolic components E_1, E_2, we can find a nonconstant harmonic function with finite Dirichelt energy and Neumann boundary condition on M by Theorem 2.5.1 . We build the cut-off function based on $\rho(x)$: fix $z\in M$, ρ is a mollification of $d_M(\cdot,z)$ such that $\rho|_{\partial B_{R_i}(z)} = R_i$ and $|\nabla\rho|\leq 2$. The cut-off $\phi_i(x)$ is equal to 1 in $B_{R_1}(z)$, it's equal to 0 outside $B_{R_i}(z)$ and equal to $\frac{R_i-\rho(x)}{R_i-R_1}$ otherwise (we may assume $B_{R_1}(z)\subset K$).

Then using ϕ_i we have as $R_i \to \infty$:

$$\frac{1}{3}\int_M \phi_i^2 |\mathrm{I\!I}|^2 |\nabla u|^2 + \frac{1}{2}\int_M \phi_i^2 |\nabla|\nabla u||^2 \leq \int_M |\nabla \phi_i|^2 |\nabla u|^2 \leq \frac{4\int_M |\nabla u|^2}{(R_i - R_1)^2} \to 0.$$

So we get that $|\mathrm{I\!I}||\nabla u| = 0 = |\nabla|\nabla u||$ over $B_{R_1}(z)$, and letting $R_1 \to \infty$ gives us the two terms vanish on M. So $|\nabla u|$ is constant, and using u has finite Dirichlet energy on M which has infinite volume, we must have $\nabla u = 0$, a contradiction since u is nonconstant. □

2.6 Free Boundary μ-bubble and Almost Linear Volume Growth

We begin with some background on Caccioppoli sets used in our setting for free boundary μ-bubbles. One can find preliminaries of Caccioppoli sets or μ-bubble in [46], [15].

Definition 2.6.1. *A measurable set Ω in a compact Riemannian manifold N^l is called a Caccioppoli set (or a set of finite perimeter) if its characteristic function χ_Ω is a function of bounded variation, i.e. the following is finite:*

$$P(\Omega) := \sup\Big\{\int_\Omega \mathrm{div}(\phi), \phi \in C_0^1(N^\circ, \mathbb{R}^l), \|\phi\|_{C^0} \leq 1\Big\},$$

We call $P(\Omega)$ the perimeter of Ω inside N(it's also equal to the $BV-$norm of χ_Ω inside N).

Using Riesz Representation theorem, the distributional derivative $\nabla(\chi_\Omega)$ is a Radon measure and we can find a Borel set (up to change of zero measure) whose topological boundary is equal the support of this measure (see [46]). We will always assume Ω is such a set and use $\partial\Omega$ to denote its reduced boundary. We note in [46] the reduced boundary is denoted as $\partial^\star\Omega$ and is contained in the topological boundary, by De Giorgi's structure theorem the $l-1$ dimensional Hausdorff measure of $\partial^\star\Omega$ is equal to $P(\Omega)$. The next lemma establishes regularity of $\partial\Omega$ for minimizers of an appropriate functional.

Consider a compact Riemannian manifold N^3 with boundary $\partial N = \partial_0 N \cup \partial_- N \cup \partial_+ N$ ($\partial_i N$ is nonempty for $i \in \{0, -, +\}$), where $\partial_- N$ and $\partial_+ N$ are disjoint and each of them intersect with $\partial_0 N$ at angles no more than $\pi/8$ inside N . We fix a smooth function $u > 0$ on N and a smooth function h on $N \setminus (\partial_- N \cup \partial_+ N)$, with $h \to \pm\infty$ on $\partial_\pm N$. We pick a regular value c_0 of h on $N \setminus (\partial_- N \cup \partial_+ N)$ and pick $\Omega_0 = h^{-1}((c_0, \infty))$. We want to find a minimizer among Caccioppoli sets for the following functional:

$$\mathcal{A}(\Omega) := \int_{\partial\Omega} u - \int_N (\chi_\Omega - \chi_{\Omega_0}) hu. \tag{2.6.1}$$

Lemma 2.6.2 (Existence of Minimizers). *There is a minimizer Ω for the above functional $\mathcal{A}$. The minimizer has smooth boundary which intersects with $\partial_0 N$ orthogonally. Also $\Omega \triangle \Omega_0$ is a compact subset in $N^\circ \cup \partial_0 N$.*

Proof. We can take Ω_0 as a candidate so the infimum value of $\mathcal{A}$ is finite. Now we take a minimizing sequence Ω_k. Using approximate identity φ_{k_j}, we have $\chi_{\hat{\Omega}} - \chi_{\Omega_0} := \chi_\Omega \star \varphi_{k_j} - \chi_{\Omega_0}$ converges in $L^p (p \geq 1)$ to $\chi_\Omega - \chi_{\Omega_0}$, together with that the $BV-$norm is lower semicontinuous with respect to L^1-norm, we can apply mollification to assume each χ_{Ω_k} has smooth boundary.

Now note that since $h \to \pm\infty$ on $\partial_\pm N$, we may assume each Ω_k contains some fixed small neighborhood $\Omega_{\tau,+}$ of $\partial_+ N$ and must not contain some fixed small neighborhood $\Omega_{\tau,-}$ of $\partial_- N$ for a $\tau > 0$ (this is proved in details in [15]) Proposition 12.

So the function $(\chi_{\Omega_k} - \chi_{\Omega_0}) hu$ is supported on the compact set $N \setminus \Omega_{\tau,\pm}$ and uniformly bounded in k, since there is some $\delta > 0$ so that $u > \delta > 0$ on $N \setminus \Omega_{\tau,\pm}$ and Ω_k is a minimizing sequence, we get that the BV-norm of Ω_k is uniformly bounded in k, and so a subsequence converge in the following sense: $\nabla(\chi_{\Omega_k})$ in the weak* sense as Radon Measures, χ_{Ω_k} in the L^1 sense, and the limit χ_{Ω_∞} is also a BV function. Therefore $\mathcal{A}(\Omega_\infty) = \lim_k \mathcal{A}(\Omega_k)$, and we found a minimizer.

We note that regularity of free boundary minimal hypersurfaces has been established by Jost-Gr̆uter [31], corresponding to the case $u = 1$ and $h = 0$ for the functional $\mathcal{A}$. For general ellipic integrand and almost minimizers, by De Philippis and Maggi [17] Theorem 1.10 or [18] Theorem 1.5, we have that $\partial\Omega$ is a $C^{1,\frac{1}{2}}$ hypersurface in N interesecting with $\partial_0 N$ orthogonally by the first variation formula given below, which also gives us the mean curvature (exists weakly *a priori*) is a smooth function, this gives smoothness of $\partial\Omega$. $\square$

We now compute the first and second variation for such minimizers.

Theorem 2.6.3. *Assume Ω is a minimizer of $\mathcal{A}$ in the settings above, we have the following first variation formula, writing $\Sigma = \partial\Omega$,*

$$\nabla_{\nu_\Sigma} u - hu + uH_\Sigma = 0 \text{ on } \Sigma, \quad \nu_{\partial\Sigma}(x) \perp T_x\partial N \quad \text{for } x \in \partial\Sigma \subset \partial_0 N,$$

and the second variation formula,

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0}(\mathcal{A}(\varphi_t(\Omega))) \\
=& \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma + |\mathrm{II}|^2 + H_\Sigma^2) + \phi^2(\Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu)) \\
&- \int_{\partial\Sigma} u\phi^2 A(\nu_\Sigma, \nu_\Sigma) \\
\le& \int_\Sigma |\nabla\phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma) + \phi^2(\Delta_N u - \Delta_\Sigma u - u\nabla_{\nu_\Sigma} h - \frac{h^2 u}{2} - \frac{u^{-1}}{2}(\nabla_{\nu_\Sigma} u)^2) \\
&- \int_{\partial\Sigma} u\phi^2 A(\nu_\Sigma, \nu_\Sigma)
\end{aligned}$$

Proof. The computation follows similarly from first and second variation formula of free boundary minimal hypersurfaces. We consider a family of diffeomorphism φ_t of N with vector field X_t, notice that if $x \in \partial N$ then $X_t \in T_x\partial N$. Let $\partial\Omega_t =: \Sigma_t$, the first variation is given as:

$$\begin{aligned}
\frac{d}{dt}(\mathcal{A}(\varphi_t(\Omega))) &= \frac{d}{dt}\int_{\partial\Omega_t} u - \frac{d}{dt}\int_{\Omega_t} hu \\
&= \int (\frac{d}{dt}u) d\mathcal{H}_{\Sigma_t} + \int u \frac{d}{dt} d\mathcal{H}_{\Sigma_t} - \int hu \frac{d}{dt} d\mathcal{H}_{\Sigma_t} \\
&= \int_{\Sigma_t} (\nabla_{X_t} u) - \int_{\Sigma_t} (hu)\langle X_t, \nu_{\partial\Omega_t}\rangle + \int_{\Sigma_t} (u \operatorname{div}_{\partial\Omega_t} X_t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma_t} (\nabla_{X_t} u) - \int_{\Sigma_t} (hu)\langle X_t, \nu_{\partial\Omega_t}\rangle + \int_{\Sigma_t} (u \operatorname{div}_{\partial\Omega_t} X_t) \\
&= \int_{\Sigma_t} (\nabla_{X_t} u) - \int_{\Sigma_t} (hu)\langle X_t, \nu_{\Sigma_t}\rangle + \int_{\Sigma_t} (u \operatorname{div}_{\Sigma_t}(X_t^\perp + X_t^\top)) \\
&= \int_{\Sigma_t} (\langle \nabla u, (X_t - X_t^\top)\rangle - hu\langle X_t, \nu_{\Sigma_t}\rangle - u\vec{H}_{\Sigma_t} \cdot X_t^\perp) + \int_{\partial\Sigma_t} u\langle X_t, \nu_{\partial\Sigma_t}\rangle \\
&= \int_{\Sigma_t} (\nabla u \cdot X_t^\perp - hu\langle X_t, \nu_{\Sigma_t}\rangle - u\vec{H}_{\Sigma_t} \cdot X_t^\perp) + \int_{\partial\Sigma_t} u\langle X_t, \nu_{\partial\Sigma_t}\rangle
\end{aligned}
$$

Recall we used the convention that mean curvature H is defined as the trace of the second fundamental form and hence $H_\Sigma = -\langle \nabla_{e_i} e_i, \nu_\Sigma\rangle = -\vec{H}_\Sigma \cdot \nu_\Sigma$. So at $t = 0$ we have $\nabla_{\nu_\Sigma} u - hu + uH_\Sigma = 0$ on Σ, and that $\nu_{\partial\Sigma}(x) \perp T_x\partial N$ for $x \in \partial\Sigma \subset \partial N$.

Now we continue with the second variation :

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} (\mathcal{A}'(\varphi_t(\Omega)) - \int_{\partial\Sigma_t} u\langle X_t, \nu_{\partial\Sigma_t}\rangle) \\
=& \int \frac{d}{dt}\Big|_{t=0} (\nabla u \cdot X_t^\perp - hu\langle X_t, \nu_{\Sigma_t}\rangle - u\vec{H}_{\Sigma_t} \cdot X_t^\perp) dvol_\Sigma \\
=& \int \phi_t \frac{d}{dt}\Big|_{t=0} (\nabla u \cdot \nu_{\Sigma_t} - hu + uH_{\Sigma_t}) dvol_\Sigma \\
=& \int_{\Sigma_t} \phi_t(\partial_t\langle \nabla u, \nu_{\Sigma_t}\rangle - \nabla_{X_t}(hu) + (\nabla_{X_t} u)H_{\Sigma_t} + u\partial_t H_{\Sigma_t}), \quad \text{at } t = 0.
\end{aligned}
$$

Since at $t = 0, \partial\Sigma \cap \partial N$ orthogonally, using the exponential map near $\partial\Sigma$, for any smooth function ϕ_t, the diffeomorphism near Σ given by $\Sigma \times (-\epsilon, \epsilon) \ni (x, t) \to \exp_x(t\nu_x)$ is admissible and produce a normal variation near Σ. We will also use $\Delta_N u - \Delta_\Sigma u = \nabla^2 u(\nu_\Sigma, \nu_\Sigma) + H_\Sigma \nabla_{\nu_\Sigma} u$:

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} (\mathcal{A}'(\varphi_t(\Omega)) - \int_{\partial\Sigma_t} u\langle X_t, \nu_{\partial\Sigma_t}\rangle) \\
=& \frac{d}{dt}\Big|_{t=0} (\mathcal{A}'(\varphi_t(\Omega))) \\
=& \int \phi_t^2(\nabla^2 u(\nu_{\Sigma_t}, \nu_{\Sigma_t}) - \nabla_{\nu_{\Sigma_t}}(hu) + H_{\Sigma_t}\nabla_{\nu_{\Sigma_t}} u) + \phi_t\langle \nabla u, \partial_t \nu_{\Sigma_t}\rangle dvol_\Sigma
\end{aligned}
$$

$$+ \int u\phi_t(-\Delta_{\Sigma_t}\phi_t - \phi_t(|\mathrm{II}_{\Sigma_t}|^2 + \mathrm{Ric}_N(\nu_{\Sigma_t}, \nu_{\Sigma_t})))dvol_\Sigma, \quad \text{at } t = 0$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - u\phi^2(|\mathrm{II}_\Sigma|^2 + \mathrm{Ric}_N(\nu_\Sigma, \nu_\Sigma)) + \phi^2\nabla^2 u(\nu_\Sigma, \nu_\Sigma) + \phi_t\langle \nabla u, \partial_t \nu_{\Sigma_t}\rangle$$
$$- \int_\Sigma \phi^2(\nabla_{\nu_\Sigma}(hu) - H_\Sigma \nabla_{\nu_\Sigma} u) + \int_\Sigma \langle \nabla_\Sigma u, \nabla_\Sigma \phi\rangle \phi - \int_{\partial\Sigma} u\phi \nabla_{\nu_{\partial\Sigma}}\phi$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - u\phi^2(|\mathrm{II}_\Sigma|^2 + \mathrm{Ric}_N(\nu_\Sigma, \nu_\Sigma)) + \phi^2(\Delta_N u - \Delta_\Sigma u) - \phi^2 \nabla_{\nu_\Sigma}(hu)$$
$$+ \int_\Sigma \phi_t \langle \nabla u, \partial_t \nu_{\Sigma_t}\rangle + \langle \nabla_\Sigma u, \nabla_\Sigma \phi\rangle \phi - \int_{\partial\Sigma} u\phi \nabla_{\nu_{\partial\Sigma}}\phi$$

Now we use that for a family of evolution of hypersurfaces using a normal vector field we have $-\nabla_\Sigma \phi = \partial_t \nu_\Sigma$ and from the Gauss equation we have $R_N = R_\Sigma + 2\,\mathrm{Ric}(\nu, \nu) + |\mathrm{II}|^2 - H_\Sigma^2$ to get,

$$\frac{d^2}{dt^2}\Big|_{t=0}(\mathcal{A}(\varphi_t(\Omega)))$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma + |\mathrm{II}|^2 + H_\Sigma^2) + \phi^2(\Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu))$$
$$- \int_{\partial\Sigma} u\phi\langle \nabla_\Sigma \phi, \nu_{\partial\Sigma}\rangle$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma + |\mathrm{II}|^2 + H_\Sigma^2) + \phi^2(\Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu))$$
$$+ \int_{\partial\Sigma} u\phi\langle \partial_t \nu_\Sigma, \nu_{\partial\Sigma}\rangle, \quad \text{at } t = 0,$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma + |\mathrm{II}|^2 + H_\Sigma^2) + \phi^2(\Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu))$$
$$+ \int_{\partial\Sigma} u\langle \nabla_{X_t} X_t, \nu_{\partial\Sigma}\rangle, \quad \text{at } t = 0,$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma + |\mathrm{II}|^2 + H_\Sigma^2) + \phi^2(\Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu))$$
$$- \int_{\partial\Sigma} u\phi^2 A(\nu_\Sigma, \nu_\Sigma)$$

We now write $|\mathrm{II}|^2 = |\mathring{\mathrm{II}}|^2 + \frac{H_\Sigma^2}{2} \geq \frac{H_\Sigma^2}{2}$ and notice that according to the first variation and $u > 0$ we have $\frac{u^{-1}}{2}(uH_\Sigma)^2 = \frac{u^{-1}}{2}(\nabla_{\nu_\Sigma} u)^2 + \frac{h^2 u}{2} - h\nabla_{\nu_\Sigma} u$, so in total:

$$
\begin{aligned}
0 \leq & \frac{d^2}{dt^2}\Big|_{t=0}(\mathcal{A}(\varphi_t(\Omega))) \\
\leq & \int_\Sigma |\nabla\phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma) + \phi^2(-\frac{3H_\Sigma^2}{4}u + \Delta_N u - \Delta_\Sigma u - \nabla_{\nu_\Sigma}(hu)) \\
& - \int_{\partial\Sigma} u\phi^2 A(\nu_\Sigma, \nu_\Sigma) \\
\leq & \int_\Sigma |\nabla\phi|^2 u - \frac{u\phi^2}{2}(R_N - R_\Sigma) + \phi^2(\Delta_N u - \Delta_\Sigma u - u\nabla_{\nu_\Sigma} h - \frac{h^2 u}{2} - \frac{u^{-1}}{2}(\nabla_{\nu_\Sigma} u)^2) \\
& - \int_{\partial\Sigma} u\phi^2 A(\nu_\Sigma, \nu_\Sigma)
\end{aligned}
$$

□

Combining the second variation of free boundary minmal hypersurface and that of $\mu-$bubble, we can produce a diameter bound as follows (see [15] for the case without boundary).

Theorem 2.6.4. *Consider $(X^4, \partial X)$ a complete manifold with $R \geq 2, H_{\partial X \geq 0}$, and $(M, \partial M) \hookrightarrow (X, \partial X)$ a two-sided stable immersed free boundary minimal hypersurface. Let N be a component of $\overline{M \setminus K}$ for some compact set K, with $\partial N = \partial_0 N \cup \partial_1 N, \partial_0 N \subset \partial M$ and $\partial_1 N \subset K$. If there is $p \in N$ with $d_N(p, \partial_1 N) > 2\pi$, then we can find a Caccioppoli set $\Omega \subset B_{2\pi}(\partial_1 N)$ whose reduced boundary is smooth, so that any component Σ of the reduced boundary $\partial^*\Omega$ will have diameter at most 2π and intersect with $\partial_0 N$ orthogonally.*

Remark 2.6.5. *For convenience we also assume $\partial_1 N \cap \partial_0 N$ at angle $\theta \in (0, \pi/8)$ within N due to similar regularity considerations as in secition 2.4. This can be arranged by purturbing N near an arbitrary small neighborhood, so will not influence the final bound for the diameter.*

Proof. We again use $\mathrm{I\!I}$ for $N \hookrightarrow X$ and A for $\partial X \hookrightarrow X$. We write ν for the outward normal of $\partial N \subset N$ (the same for $\partial X \subset X$). For any variation φ_t of $(N, \partial N)$ compactly supported away from $\partial_1 N$, writing $\frac{d}{dt}\big|_{t=0}\varphi_t = f\nu_N$, with ν_N a unit normal of $N \hookrightarrow X$, we have by the second variation formula for stable free boundary minimal hypersurfaces:

$$0 \leq \frac{d^2}{dt^2}\Big|_{t=0}\text{Area}(\varphi_t(N)) = \int_N |\nabla_N f|^2 - (|\mathrm{II}|^2 + \text{Ric}(\nu_N, \nu_N))f^2 - \int_{\partial_0 N} A(\nu_N, \nu_N) f^2.$$

Integration by parts gives us,

$$0 \leq \int_N -(f\Delta_N f + |\mathrm{II}|^2 f^2 + \text{Ric}(\nu_N, \nu_N) f^2) + \int_{\partial_0 N} f(\nabla_\nu f - A(\nu_N, \nu_N) f).$$

We denote the first eigenvalue as:

$$\lambda_1(N) = \min_S \frac{\int_N -(f\Delta_N f + |\mathrm{II}_N|^2 f^2 + \text{Ric}(\nu_N, \nu_N) f^2)}{\int_N f^2},$$

where $S = \{f \neq 0, f|_{\partial_1 N} = 0 \text{ and } \nabla_\nu f - A^X(\nu_N, \nu_N) f = 0 \text{ on } \partial_0 N\}$ and each test function f is taken to be compactly supported and $\lambda_1(N)$ is well-defined by domain monotonicity property for compact sets ($\lambda_1(B_1) < \lambda_1(B_2)$ if $\overline{B_1} \subset B_2$) from Fischer-Colbrie and Schoen [20] for example.

We first show that there is a C^3 positive solution to $\Delta_N f + (|\mathrm{II}|^2 + \text{Ric}(\nu_N, \nu_N)) f = 0$ and $\nabla_\nu f - A(\nu_N, \nu_N) f = 0$ along $\partial_0 N$.

We consider the following problem over a compact exhaustion (Ω_l) of N, each containing the boundary $\partial_1 N$:

$$\begin{aligned}
(\Delta_N + |\mathrm{II}_N|^2 + \text{Ric}(\nu_N, \nu_N)) f = 0, &\quad \Omega_l^\circ \\
\nabla_\nu f - A(\nu_N, \nu_N) f = 0, &\quad \partial_0 N \cap \Omega_l \\
f = 0, &\quad \partial^* \Omega_l \\
f = 1, &\quad \partial_1 N.
\end{aligned}$$

By domain monotonicity we have $\lambda_1(\Omega_l) > 0$ for each Ω_l so the above problem has a unique solution in H^1 and via interior and boundary regularity, we have each solution v_l is $C^3(\overline{\Omega_l})$. We claim that each $v_l > 0$ on Ω_l, by Hopf Lemma ([23] Lemma

3.4), it's enough to show $v_l \geq 0$.

Now assume $\{v_l < 0\} \neq \emptyset$, we write $v_l = v^+ - v^-$, we have that $(\Delta_N + |\mathrm{II}_N|^2 + \mathrm{Ric}(\nu_N, \nu_N))v^- \geq 0$ and since $v \in H^2$ we get that on $\partial_0 N$, v^- is either 0 or has $\nabla_\nu v^- + A(\eta, \eta)v^- = 0$ in H^1 sense. Now using v^- as a test function we get:

$$0 \geq \int_N -(v^-\Delta_N v^- + |\mathrm{II}|^2(v^-)^2 + \mathrm{Ric}(\nu_N, \nu_N)(v^-)^2) + \int_{\partial_0 N} v^-(\nabla_\nu v^- - A(\nu_N, \nu_N)v^-),$$

a contradiction to $\lambda_1(\Omega_l) > 0$.

Now we have that $v_l > 0$ and $v_l|_{\partial_1 N} = 1$ then we can proceed as in [20], Harnack inequality gives v_l subsequentially converge in C^2_{loc} to a nonzero function on N, with $u > 0$ on N°, $u|_{\partial_1 N} = 1$ and,

$$(\Delta_N + Ric_X(\nu_N, \nu_N) + |\mathrm{II}_N|^2)u = 0 \quad \text{on } N^\circ, \quad \nabla^N_\nu u - A(\nu_N, \nu_N)u = 0 \quad \text{on } \partial_0 N. \tag{2.6.2}$$

Now we follow Chodosh-Li-Stryer [15] and apply the free boundary μ bubble to the above u and a proper h.

Consider a mollification of $d(\cdot, \partial_1 N)$ with Lipschitz constant less than 2, denoted as ρ_0, we may assume that $\rho_0(x) = 0$ for all $x \in \partial_1 N$, and the level set $\{\rho_0(x) = 2\pi\}$ is a smooth submanifold in N.

Define $\Omega_1 := \{x \in N, 0 < \rho_0 < 2\pi\}$, $\Omega_0 := \{0 < \rho_0 < \pi\}$, and set

$$h(x) := -\tan\left(\frac{1}{2}\rho_0(x) - \frac{\pi}{2}\right) = -\tan(\rho(x)).$$

We solve the $\mu-$bubble problem among Caccioppoli sets whose symmetric difference with Ω_0 is compact in Ω_1, i.e. we minimize the functional $\mathcal{A}(\Omega)$ using the given h and $u > 0$ obtained above. We obtain a minimizer Ω, and for any component Σ of $\partial^*\Omega$, we have $\partial\Sigma \cap \partial_0 N$ orthogonally and from the second variation formula in Theorem 2.6.3 we get for any compactly supported smooth function ϕ on Σ (Lemma 15 of [13]),

$$0 \leq \int_\Sigma |\nabla_\Sigma \phi|^2 u - \frac{1}{2}(R_N - 1 - R_\Sigma)\phi^2 u + (\Delta_N u - \Delta_\Sigma u)\phi^2 - \frac{1}{2u}(\nabla_{\nu_\Sigma} u)^2\phi^2$$

$$\int_{\Sigma} -\frac{1}{2}(1+h^2+2\nabla_{\nu_\Sigma} h)\phi^2 u - \int_{\partial\Sigma} A(\nu_\Sigma,\nu_\Sigma)\phi^2 u,$$

and now we have that $1+h^2+2\nabla_{\nu_\Sigma} h \geq 1+\tan^2(\rho)-\sec^2(\rho) = 1-1 = 0$. So in total we have:

$$\begin{aligned}0 \leq \int_{\Sigma} &|\nabla_\Sigma \phi|^2 u - \frac{1}{2}(R_N - 1 - R_\Sigma)\phi^2 u + (\Delta_N u - \Delta_\Sigma u)\phi^2 - \frac{1}{2u}(\nabla_{\nu_\Sigma} u)^2\phi^2 \\ &- \int_{\partial\Sigma} A(\nu_\Sigma,\nu_\Sigma)\phi^2 u.\end{aligned}$$

We can plug in the equation (2.6.2) for u, using $R_g \geq 2$ and Gauss Equation $R_X = R_N + 2\operatorname{Ric}_X(\nu_N,\nu_N) + |\mathrm{II}_N|^2 - H_N^2$ to get:

$$\begin{aligned}0 &\leq \int_{\Sigma} |\nabla_\Sigma \phi|^2 u - \frac{1}{2}(1-R_\Sigma)\phi^2 u - \Delta_\Sigma u\phi^2 - \frac{1}{2u}(\nabla_{\nu_\Sigma} u)^2\phi^2 - \int_{\partial\Sigma} A(\nu_\Sigma,\nu_\Sigma)\phi^2 u \\ &\leq \int_{\Sigma} |\nabla_\Sigma \phi|^2 u - (\frac{1}{2} - K_\Sigma)\phi^2 u - \Delta_\Sigma u\phi^2 - \int_{\partial\Sigma} A(\nu_\Sigma,\nu_\Sigma)\phi^2 u \\ &\leq \int_{\Sigma} -\operatorname{div}(u\nabla_\Sigma\phi)\phi - (\frac{1}{2} - K_\Sigma)\phi^2 u - \Delta_\Sigma u\phi^2 - \int_{\partial\Sigma} (A(\nu_\Sigma,\nu_\Sigma)\phi - \langle\nabla_\Sigma\phi,\eta\rangle)\phi u\end{aligned}$$

By the same argument we used above to obtain u, we can find a function w with $A(\nu_\Sigma,\nu_\Sigma)w - \langle\nabla_\Sigma w,\eta\rangle = 0$ so that on Σ,

$$\operatorname{div}(u\nabla_\Sigma w) + (\frac{1}{2} - K_\Sigma)uw + w\Delta_\Sigma u = 0$$

We let $f = uw$ and by combining the equation above we have over Σ:

$$\begin{aligned}\Delta_\Sigma f &= w\Delta_\Sigma u + \operatorname{div}_\Sigma(u\nabla_\Sigma w) + \nabla_\Sigma u\cdot\nabla_\Sigma w \\ &= -(\frac{1}{2} - K_\Sigma)uw + \nabla u\cdot\nabla w \\ &= -(\frac{1}{2} - K_\Sigma)uw + \frac{1}{2uw}(|\nabla f|^2 - u|\nabla w|^2 - w|\nabla u|^2) \\ &\leq -(\frac{1}{2} - K_\Sigma)f + \frac{1}{2f}|\nabla f|^2.\end{aligned}$$

Lemma 17 in Chodosh-Li [13] also holds under the following condition (a short proof is obtained in lemma 2.6.6 below):

$$\partial_\eta f = u\partial_\eta w + w\partial_\eta u = A(\nu_\Sigma, \nu_\Sigma)uw + A(\nu_N, \nu_N)uw = H_{\partial X} f - k_{\partial\Sigma} f \geq -k_{\partial\Sigma} f.$$

So $\operatorname{diam}(\Sigma) \leq 2\pi$. □

Lemma 2.6.6. *If $(\Sigma^2, \partial\Sigma, g)$ is a compact Riemannian manifold with Gauss curvature K_Σ and,*

$$\Delta_\Sigma \lambda \leq -(K_0 - K_\Sigma)\lambda + \frac{|\nabla_\Sigma \lambda|^2}{2\lambda}, \quad \nabla_\eta \lambda + k_{\partial\Sigma}\lambda \geq 0 \tag{2.6.3}$$

for some smooth $\lambda > 0$, η the outward unit normal of $\partial\Sigma \subset \Sigma$, $k_{\partial\Sigma}$ the corresponding geodesic curvature and $K_0 \in (0, \infty)$. Then $diam_g \Sigma \leq \sqrt{\frac{2}{K_0}}\pi$.

Proof. We follow the proof of Lemma 16 and Lemma 17 in [13] and track the sign of the boundary terms carefully. If not, then we can find a free boundary curve $\gamma : [a, b] \to \Sigma$ with $\partial\gamma \subset \partial\Sigma$ and the following (from Proposition 15 in [13]), take $u = \lambda$ and $\psi^2 u = 1$,

$$\begin{aligned}
0 \leq & \int_\gamma |\nabla_\gamma \psi|^2 u - \frac{1}{2}(R_\Sigma - 2K_0)\psi^2 u + (\Delta_\Sigma u - \Delta_\gamma u)\psi^2 - \frac{1}{2}(2K_0 + h^2 + 2\nabla_{\nu_\gamma} h)\psi^2 u \\
& - \int_\gamma \frac{|\nabla_{\nu_\gamma} u|^2}{2u^2} - \int_{\partial\gamma} \mathrm{II}_{\partial\Sigma}(\nu_\gamma, \nu_\gamma)\psi^2 u \\
= & \int_\gamma \frac{1}{4u^2}|\nabla_\gamma u|^2 - \frac{1}{2}(R_\Sigma - 2K_0) + u^{-1}(\Delta_\Sigma u - \Delta_\gamma u) - \frac{1}{2}(2K_0 + h^2 + 2\nabla_{\nu_\gamma} h) \\
& - \int_\gamma \frac{|\nabla_{\nu_\gamma} u|^2}{2u^2} - \int_{\partial\gamma} \mathrm{II}_{\partial\Sigma}(\nu_\gamma, \nu_\gamma) \\
\overset{(\star 1)}{<} & \int_\gamma \frac{1}{4u^2}|\nabla_\gamma u|^2 + \frac{|\nabla_\Sigma u|^2 - |\nabla_{\nu_\gamma} u|^2}{2u^2} - u^{-1}\Delta_\gamma u - \int_{\partial\gamma} \mathrm{II}_{\partial\Sigma}(\nu_\gamma, \nu_\gamma) \\
\overset{(\star 2)}{=} & \int_\gamma \frac{3}{4u^2}|\nabla_\gamma u|^2 + \nabla_\gamma(u^{-1})\nabla_\gamma u - \int_{\partial\gamma} u^{-1}\nabla_{\nu_{\partial\gamma}} u + \mathrm{II}_{\partial\Sigma}(\nu_\gamma, \nu_\gamma) \\
\overset{(\star 3)}{=} & \int_\gamma \frac{-1}{4u^2}|\nabla_\gamma u|^2 - \int_{\partial\gamma} u^{-1}\nabla_\eta u + k_{\partial\Sigma} \leq 0,
\end{aligned}$$

where in $(\star 1)$ we used the assumption (2.6.3) and that $(K_0 + \frac{1}{2}h^2 + \nabla_{\nu_\gamma} h) > 0$ as in Lemma 16 of [13]; in $(\star 2)$ we used integration by parts; in $(\star 3)$ we used $\nu_{\partial\gamma} = \eta$ by free boundary. The strict inequality gives a contradiction. □

Theorem 2.6.7 (Almost Linear Growth of An End). *Let $(X^4, \partial X)$ be a complete manifold with weakly bounded geometry, $H_{\partial X} \geq 0, \mathrm{Ric}_2 \geq 0$ and $R_g \geq 2$. Let $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ be a complete simply connected two-sided stable free boundary minimal immersion. Let $(E_k)_{k\in\mathbb{N}}$ be an end of M given by $E_k = M \setminus B_{kL}(x)$ for some fixed point $x \in M$ and let $M_k := E_k \cap \overline{B_{(k+1)L}(x)}$, here $L = 20\pi$ (determined by the constant in the lemma above). Then there is a constant $C_0 = C(X, L)$ and k_0 such that for $k \geq k_0$,*

$$Vol_M(M_k) \leq C_0.$$

Proof. The proof that there is a large k_0 so that for all $k \geq k_0$, M_k is connected is the same as [15] Proposition 3.2 (this uses the simply-connectedness). For each E_k we can purturb the boundary so that it intersects with ∂M with an interior angle $\theta \in (0, \pi/8)$ and we can apply Theorem (2.6.4) to $E_k \hookrightarrow X$, so we obtain $\Omega_k \subset B_{\frac{L}{2}}(\partial E_k)$. Also with the same proof as [15] Lemma 5.4, there is some component Σ_k of $\partial\Omega_k$ that separates ∂E_k and ∂E_{k+1}, then Theorem (2.6.4) implies that $\mathrm{diam}(\Sigma_k) \leq c$ for $(c = 2\pi)$ and $\mathrm{diam}(M_k) \leq 4L + c$. We can show this last inequality by taking any two points z_1, z_2 in M_k, for each z_i there is a minimizing path connecting x and z_i and intersecting Σ_k at some point y_i, the arc connecting y_i, z_i is at most $2L$ and combining with $\mathrm{diam}(\Sigma_k) \leq c$ we get $d(z_1, z_2) \leq 4L + c$.

Now by curvature estimates Lemma 2.3.1 we can apply the volume control Lemma 2.3.3, to get a constant $C_0 = C(X, g, L, c)$ such that,

$$\mathrm{Vol}(B_{4L+c}(p)) \leq C_0,$$

for all $p \in M$. Since $\mathrm{diam}(M_k) \leq 4L + c$, we get $\mathrm{Vol}(M_k) \leq C_0$ as desired. □

2.7 Proof of Main Theorem

Now we are ready to prove the main theorem. We first explain some set up.

We first assume M is simply connected and has infinte volume (otherwise the proof is the same as assuming M is compact as described in the introduction), and by section 2.5 we know M has at most 1 nonparabolic end $(E_k)_{k\in\mathbb{N}}$ which we can apply Theorem 2.6.7 to obtain M_k and k_0, L, c following the notation in Theorem (2.6.7).

We write M as a decomposition of the following components, fixing $x \in M$ and write $B_R(x)$ as B_R,

$$\begin{aligned} M &= \overline{B_{k_0L}} \cup E_{k_0} \cup (M \setminus (B_{k_0L} \cup E_{k_0})) \\ &=: \overline{B_{k_0L}} \cup E_{k_0} \cup P_{k_0} \end{aligned}$$

We also have inductively,for each $i \geq 1$:

$$\begin{aligned} E_{k_0} &= M_{k_0} \cup P_{k_0+1} \cup E_{k_0+1} \\ &= M_{k_0} \cup P_{k_0+1} \cup (M_{k_0+1} \cup P_{k_0+2} \cup E_{k_0+2}) \\ &= \left(\bigcup_{k=k_0}^{k_0+i-1} M_k \right) \cup \left(\bigcup_{k=k_0+1}^{k_0+i} P_k \right) \cup E_{k_0+i} \end{aligned}$$

where each P_k when $k > k_0$ is defined as $E_k \setminus (E_{k+1} \cup B_{(k+1)L})$, and each component of P_k for $k \geq k_0$ is parabolic.

We restate the main theorem for convenience of reader:

Theorem 2.7.1. *Let $(X^4, \partial X)$ be complete with $R_g \geq 2$, $Ric_2 \geq 0$, weakly bounded geometry and weakly convex boundary. Then any complete stable two-sided free boundary minimal hypersurface $(M^3, \partial M)$ is totally geodesic and $Ric(\eta, \eta) = 0$ along M and $A(\eta, \eta) = 0$ along ∂M, for η a choice of normal bundle over M.*

Proof. Following the set up above, fix $x \in M$, $i \geq 1$ and obtain k_0, L, c, E_k, M_k, P_k.

For each $k \geq k_0$, P_k is made of disjoint parabolic components. P_{k_1} and P_{k_2} are also disjoint if $k_1 \neq k_2$. So we can apply Lemma (2.4.8) to each of these component, and obtain a compactly supported function u_k on each P_k, with $\int_{P_k} |\nabla u_k|^2 < \frac{1}{i^2}$ and with the boundary condition $u_k|_{\partial P_k \setminus \partial M} = 1, \nabla_\nu u|_{\partial M \cap P_k} = 0$.

We let ρ a mollification of the distance function to x, with $|\nabla \rho| \leq 2$ and

$$\rho|_{\partial E_k} = kL, \rho|_{\partial M_k \setminus \partial E_k} = (k+1)L.$$

Consider $\phi(x) = \frac{(k_0+i)L-x}{iL}$, then we can define a compactly supported Lipschitz function f_i as follows. When $x \in \overline{M_k}$ for some $k_0 \le k \le k_0 + i - 1$, then $f_i(x) = \phi(\rho(x))$, and when $x \in \overline{P_k}$ for some $k_0 \le k \le k_0 + i$ we define $f(x) = \phi(kL)u_k$. One can check that this definition agrees on the intersection, and we can define $f(x) = 1$ when $x \in \overline{B_{k_0 L}}$, and $f(x) = 0$ when $x \in E_{k_0+i}$. Now we can apply this test function into the stability inequality for free boundary minimal hypersurface, together with $A \ge 0$:

$$\begin{aligned}
\int_M (\mathrm{Ric}(\eta,\eta) + |\mathrm{I\!I}|^2) f_i^2 &\le \int_M |\nabla f_i|^2 - \int_{\partial M} A(\eta,\eta) f^2 \\
&\le \sum_{k=k_0}^{k_0+i-1} \int_{M_k} \phi'(\rho)^2 |\nabla \rho|^2 + \sum_{k=k_0}^{k_0+i} \phi^2(kL) \int_{P_k} |\nabla u_k|^2 \\
&\le \frac{4iC_0}{i^2 L^2} + \frac{i+1}{i^2} \le \frac{C'}{i} \to 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Since $f_i \to 1$ on M as we let $i \to \infty$, we get that everywhere on M, $\mathrm{Ric}(\eta,\eta) = 0$ and $\mathrm{I\!I} = 0$, and $A(\eta,\eta) = 0$ along ∂M. □

We note that until the last step, $A_2 \ge 0$ is sufficient. We now provide a counterexample to Theorem 2.7.1 if one replace $A \ge 0$ by $A_2 \ge 0$.

Consider $\mathbb{S}^4 \subset \mathbb{R}^5$ with induced metric, and any closed curve $\gamma \subset \mathbb{S}^4$, we look at the intrinsic neighborhood $X = B_\epsilon(\gamma) := \{x \in \mathbb{S}^4, d(x,\gamma) \le \epsilon\}$. We can choose γ so that $A_2 \ge 0$ everywhere but $A(e_1, e_1) < 0$ for some nonzero e_1 at a point in X. We can minimize area among all hypersurfaces with (nonempty) boundary and nontrivial homology class contained in ∂X, then we have a stable free boundary minimal immersion.

Chapter 3

Uniformly Mean Convex Manifolds with Non-negative Scalar Curvature

In chapter 2 we studied free boundary minimal hypersurfaces in manifolds with uniformly positive curvature and mean convex boundary, for example a spherical cap. In chapter 4 we will study stable free boundary minimal hypersurfaces in $\mathbb{B}^4$, which has non-negative sectional curvature but uniformly mean convex boundary. This means we need to find analogous geometric control for such manifolds.

In this chapter we use stable generalized capillary surfaces (analogous to the μ-bubble construction) to study manifolds with strictly mean convex boundary and nonnegative scalar curvature. We give an obstruction to filling 2-manifolds by such 3-manifolds based on the Urysohn width. We also obtain a bandwidth estimate and establish other geometric properties of such manifolds.

Results in this chapter come from [66].

3.1 Introduction

In [27], Gromov asked the question of finding sufficient conditions to allow or disallow filling in a given Riemannian manifold Y^n as the boundary of a Riemannian manifold X^{n+1} with nonnegative scalar curvature. Can the mean curvature of $Y = \partial X$ prove enough influence so that the we cannot prescribe certain geometry properties on X?

In this paper we are interested in the case when $n = 2$. If Y is a connected orientable closed surface with positive Gaussian curvature, then there is an isometric embedding of Y into $\mathbb{R}^3$ as a strictly convex surface, we denote the mean curvature of this embedding as H_0 (such embedding is unique up to isometry of $\mathbb{R}^3$). Using this, Shi-Tam [58] proved that if there is some (X, g) with nonnegative scalar curvature filling such a Y with positive mean curvature H, then

$$\int_Y H d\sigma \leq \int_Y H_0 d\sigma,$$

where $d\sigma$ is the volume form induced from the metric g. Moreover, equality holds if and only if X is a domain in $\mathbb{R}^3$. This result gives a positive answer to the question in the first paragraph.

In this paper, we give another answer in the following theorem using (generalized) capillary surfaces. As an example, if a 3-manifold fills $\mathbb{S}^2$ with $H \geq 2$ and has nonnegative scalar curvature, then the first Urysohn width of $\mathbb{S}^2$ in the induced metric is no more than 4.5π (take $a_0 = \frac{2}{3}, d_0 = \frac{3\pi}{4}$). Recent results in upper bound of Urysohn width was also obtained in [63], [45].

Theorem 3.1.1. *If $(N^3, \partial N, g)$ is a simply connected complete manifold such that $R_g \geq 0$ and $H_{\partial N} \geq \frac{\pi}{d_0} + a_0$, then the first Urysohn width of ∂N (with respect to the induced metric g) is bounded: $U_1(\partial N) \leq 4d_0 + \frac{\pi}{a_0}$.*

In another direction, Gromov proved the following bandwidth estimate, here we denote T^{n-1} as the $n-1$ dimensional torus. The theorem can be proved using the idea of μ-bubbles [28].

Theorem (Bandwidth Estimate, [26]). *Let $M_0 = T^{n-1} \times [-1, 1]$, with $\partial_\pm M_0 = T^{n-1} \times \{\pm 1\}$, if a Riemannian manifold $(M^n, g), 2 \leq n \leq 7$ admits a continuous map $f : (M, \partial_\pm M) \to (M_0, \partial_\pm M_0)$ with nonzero degree and scalar curvature bounded from below $R_g \geq n(n-1)$, then the distance of $\partial_+ M$ and $\partial_- M$ is bounded: $dist_g(\partial_+ M, \partial_- M) \leq \frac{2\pi}{n}$.*

Using μ-bubbles, that is, studying stable hypersurfaces with prescribed mean curvature in a manifold with positive scalar curvature (PSC) has given fruitful results in recent years (see for example [40] [69][13][15]). In these works, the fact that the scalar curvature of the manifold has to obtain a strictly positive lower bound is crucial. On the other hand, for manifolds with boundary, to constrain a minimizer of a generalized area functional, we can prescribe both mean curvature and the angle of intersection along the boundary. In particular, mean convexity assumption is helpful to constrain capillary surfaces. This allows us to relax the assumption of PSC when studying manifolds with boundary.

Gromov studied a band with PSC of the form $\Sigma^2 \times [-1,1]$, for a closed surface Σ with $\chi_\Sigma \leq 0$ (for example the torus). Our model example in case of surface with boundary would be $M_0 = (\Sigma_0, \partial\Sigma_0) \times [-1,1]$, with a Riemannian metric such that $R_{M_0} \geq 0$, and the boundary $\partial_0 M$ is strictly mean convex. We will use capillary surfaces in M_0 as an analogy to the μ-bubbles in Gromov's bandwidth estimate.

Theorem 3.1.2 (Bandwidth estimate)**.** *Consider a compact 3-manifold $(M, \partial M)$ with the following decomposition (see Figure 1),*

$$M = (\Sigma_0, \partial\Sigma_0) \times [-1,1], \qquad \partial M = \partial_0 M \cup \partial_\pm M,$$
$$\partial_\pm M = \Sigma_0 \times \{\pm 1\}, \qquad \partial_0 M = \partial\Sigma_0 \times (-1,1).$$

Here Σ_0 is an orientable surface with boundary and Euler characteristic $\chi(\Sigma_0) \leq 0$.

We denote the scalar curvature of M by R_M, mean curvature of $\partial_0 M$ by H_0, the top $\partial_+ M$ as S_+, the bottom $\partial_- M$ as S_-. If M has mean convex boundary ∂M ($H_{\partial M} > 0$), $R_M \geq 0, H_0 \geq 1$, then $d_{\partial M}(\partial S_+, \partial S_-) \leq \pi$, in particular, $d_M(S_+, S_-) \leq \pi$.

In [52], Ros and Souam studied constant mean curvature surfaces which intersect the ambient boundary at a constant angle, called capillary surfaces. In the second variation formula for generalized capillary surfaces, instead of constant angle, we allow the prescribed angle to vary. For manifolds with nonnegative scalar curvature and strictly mean convex boundary, we are able to gain rigidity results for 2-manifolds (Theorem 3.1.3), and exhaustion of the boundary for 3-manifolds (Theorem 3.1.4).

Theorem 3.1.3. *Consider a complete connected Riemannian manifold Σ^2 with boundary, such that $R_\Sigma \geq 0, k_{\partial\Sigma} \geq 1$, then $\partial\Sigma$ is connected with length no more than 2π;*

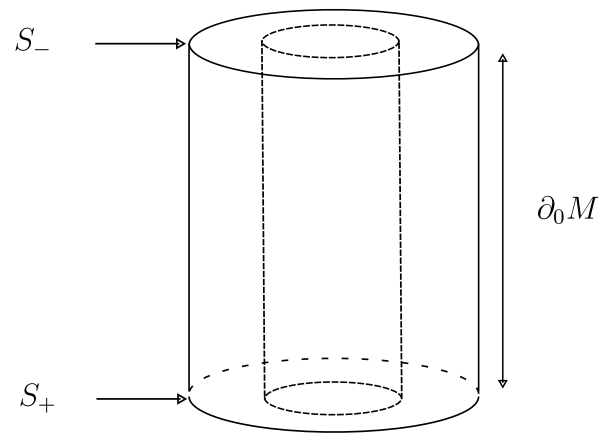

Figure 3.1: An example of M when Σ_0 is an annulus.

for any $x \in \Sigma$, $d_\Sigma(x, \partial\Sigma) \leq 1$, *so* Σ *is compact.* Σ *is a topological disk. If* $|\partial\Sigma| = 2\pi$, *then* Σ *is isometric to a flat unit disk in* $\mathbb{R}^2$.

The fact that every point in this manifold must be at most distance 1 away from the boundary follows from Li-Nguyen [43] and Li [41]. Furthermore, Li [41] conjectured the following: A complete Riemannian manifold M^n with nonnegative Ricci curvature and $A_{\partial M} \geq 1$ must be compact ($A_{\partial M}$ is the second fundamental form of the boundary). The case for surfaces has been known (see [22] for general Alexandov spaces set up). This theorem gives an alternative variational proof to the 2-dimensional case of this conjecture. We note in Theorem 3.1.3 we do not assume orientability or embeddedness. In the case of $M^n \subset \mathbb{R}^{n+1}$, a strictly convex hypersurface bounding a region with pinched second fundamental form, Hamilton [35] showed that M must be compact.

Contrary to the 2-dimensional case, for a connected 3-manifold with nonnegative scalar curvature and $H_{\partial M} \geq 2$ ($H_{\partial M}$ is mean curvature of the boundary), its boundary might not be connected or compact. But we can obtain an exhaustion of the boundary as below, and show that the boundary has linear volume growth under further topological assumptions (see Corollary 3.1.5).

Theorem 3.1.4. *Let* $(M^3, \partial M)$ *be a complete connected Riemannian manifold with nonnegative scalar curvature, and* ∂M *is connected non-compact with* $H_{\partial M} \geq 2$.

We can find an exhaustion of the boundary $\partial M = \cup_k Z_k$, with $\partial Z_k = \partial\Sigma'_k$, where Σ'_k is a union of finitely many capillary surface of disk type in M. Each component Σ_k of Σ'_k has bounded boundary length: $|\partial\Sigma_k| \leq 2\pi$, and $d_{\Sigma_k}(x, \partial\Sigma_k) \leq 2$ for any $x \in \Sigma_k$.

Furthermore, assume either ∂M or M is simply connected. Let E_k be an unbounded component of $\partial M \setminus Z_k$ such that $E_{k+1} \subset E_k$ (i.e.$(E_k)_{k=1}^{\infty}$ is an end of ∂M), then $\overline{E_k} \setminus E_{k+1}$ is connected and $\sup_k diam_{\partial M}(\overline{E_k} \setminus E_{k+1}) \leq 5\pi$.

Using Theorem 3.1.4 and simply-connectedness, if ∂M has weakly bounded geometry, we can apply section 3 in [65] to get the following corollary.

Corollary 3.1.5. *Let $(M^3, \partial M)$ be a complete connected Riemannian manifold with nonnegative scalar curvature. Assume either ∂M or M is simply connected. If ∂M is uniformly mean convex and has weakly bounded geometry, then each end of ∂M has linear volume growth. In particular, if ∂M has finitely many ends, then ∂M has linear volume growth.*

3.2 Result for Surfaces

Throughout this paper, if $(N^{k+1}, \partial N, g), k \geq 1$, is a Riemannian manifold with boundary, we denote the second fundamental form of $\partial N \subset N$ with respect to the outward pointing unit normal ν as $\mathbb{I}(X, Y) = -\langle \nabla_X Y, \nu \rangle$ for vector fields X, Y tangent to ∂N. Then the scalar-valued mean curvature is written as $H = \sum_{i=1}^k -\langle \nabla_{e_i} e_i, \nu \rangle$ for an orthonormal basis $(e_i)_i^k$ of $T_p\partial N$ at some point $p \in \partial N$. In this convention, the boundary of the unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ has positive mean curvature, $H_{\partial \mathbb{B}^3} = 2$.

We first prove Theorem 3.1.3, using "capillary curves" in surfaces of nonnegative scalar curvature and strictly convex boundary.

Proof of Theorem 3.1.3. Consider Σ, with $R_\Sigma \geq 0$ and geodesic curvature $k_{\partial\Sigma} \geq 1$.

If Σ is compact, the $d_\Sigma(x, \partial\Sigma) \leq 1$ part follows directly from [43, Proposition 2.1] and [41, Theorem 1.1]. Note that from this result we know that Σ is compact if and only if $\partial\Sigma$ is compact.

Step 1. If we know that Σ is compact, we can show that $\partial\Sigma$ is connected and the length of $\partial\Sigma$ is no more than 2π.

We apply Gauss Bonnet theorem to get, denoting $\frac{1}{2}R_\Sigma = K_\Sigma$:

$$2\pi \geq 2\pi\chi = \int_\Sigma K_\Sigma + \int_{\partial\Sigma} k_{\partial\Sigma} \geq |\partial\Sigma| > 0. \tag{3.2.1}$$

Here we used Σ has at least one boundary component, so $\chi_\Sigma \leq 1$. And if $\partial\Sigma$ has more than 1 boundary component then we would get $\chi_\Sigma \leq 0$, a contradiction. So $\partial\Sigma$ is connected with $|\partial\Sigma| \leq 2\pi$, and $\chi(\Sigma) > 0$, so $\chi_\Sigma = 1$, and Σ is a topological disk.

Step 2. We now look at the rigidity statement under the assumption that Σ is compact (and $\partial\Sigma$ is connected by Step 1).

If $K \geq 0, k \geq 1$ and $|\partial\Sigma| = 2\pi$, then from (3.2.1) we must have $K = 0, k = 1$ everywhere. So Σ is locally isometric to the flat unit disk $\mathbb{D} = \{(x_1, x_2), x_1^2 + x_2^2 \leq 1\}$, we show that the local isometry can be extended globally.

We denote the unit speed loop of $\partial\Sigma$ as $\gamma : [0, 2\pi] \to \partial\mathbb{D}$, and write $\gamma(0) = \gamma(2\pi) = p$. Then by choosing ϵ small enough, we can find an isometry for each $t \in [0, 2\pi]$ with $\Psi_t : B_\epsilon(\gamma(t)) \to \mathbb{D}$. Combining Ψ_t with a rotation of $\mathbb{D}$ around origin, we can patch up these isometry Ψ_t to get a global isometry Ψ_0 from $B_\epsilon(\partial\Sigma)$ to a neighborhood of $\partial\mathbb{D}$ in $\mathbb{D}$.

We now define a global isometry Ψ from Σ to $\mathbb{D}$ as follows, for $x \in B_\epsilon(\partial\Sigma)$, $\Psi(x) = \Psi_0(x)$. Now fix $p \in \partial\Sigma$ and for any $q \in \Sigma \setminus B_\epsilon(\partial\Sigma)$, for any path $s(t) : (0, 1] \to \Sigma^\circ$ with $s(0) = p, s(1) = q$. We cover $s(t)$ with interior balls each of which can be mapped isometrically into an interior ball in $\mathbb{D}$, in a way that agrees on the overlap (see [39, Theorem 12.4]). Then applying in [39, Corollary 12.3], we have obtained a global isometry from Σ to $\mathbb{D}$.

Step 3. We now show $\partial\Sigma$ is connected (if Σ is non-compact).

If not, then there are at least two components $\partial_1\Sigma, \partial_2\Sigma$ of $\partial\Sigma$. Let's first assume that $\alpha = \inf\{d(x, y), x \in \partial_1\Sigma, y \in \partial_2\Sigma\}$ can be obtained by a stable free boundary geodesic $\beta_0(t)$. Let T be a unit normal vector field of β_0, notice that along $\partial\Sigma$, T is tangent to $\partial\Sigma$, this vector field generates a local variation of β_0 denoted by $\beta_s(t)$.

Then the second variation formula of length $L(\beta_s)$ implies,

$$0 \leq \frac{d^2}{ds^2}\bigg|_{s=0} L(\beta_s) = \int_{\beta_0} -\frac{R_\Sigma}{2} + \int_{\partial\beta_0} \langle \nabla_T T, -\beta_0' \rangle$$
$$\leq \langle \nabla_T T, -\beta_0'(0) \rangle + \langle \nabla_T T, \beta_0'(\alpha) \rangle \leq -2 < 0,$$

where we used convexity at the end points, i.e. $\langle \nabla_T T, \beta_0'(0) \rangle = k_{\partial\Sigma}(\beta_0(0)) \geq 1$ and $\langle \nabla_T T, \beta_0'(\alpha) \rangle = -k_{\partial\Sigma}(\beta_0(\alpha)) \leq -1$; this is a contradiction.

Now we look at the case α can't be realized by a stable free boundary geodesic. In this case, at least one of the component is not compact. Assume $\partial_1\Sigma$ is non-compact, then we can try to find a minimizing geodesic line via points on $\partial_1\Sigma$. To elaborate, consider the universal cover of Σ named $\tilde{\Sigma}$, then we have again (at least) two components of the boundary $\partial\tilde{\Sigma}$, denoted $\partial_1\tilde{\Sigma}, \partial_2\tilde{\Sigma}$. Assume $\partial_1\tilde{\Sigma}$ is not compact. We denote $\partial_1\tilde{\Sigma} \cong \mathbb{R}$ as $\gamma(t)$, with $|\gamma'(t)| = 1$, $\gamma(0) = q$, $\gamma(\pm n) = \pm q_n$. Consider the minimizing geodesic in $\tilde{\Sigma}$ from $+q_n$ to $-q_n$ denoted $l_n(t)$, which cannot touch the boundary except at end points due to strict convexity.

We fix a point $p \in \partial_2\tilde{\Sigma}$ and a minimizing path from p to q, denoted by $\overline{pq}(t)$. Now any minimizing geodesic in $\tilde{\Sigma}$ between $\pm q_n$ must intersect $\overline{pq}$. Indeed, if there is a minimizing geodesic l between $\pm q_n$ and $l \cap \overline{pq} = \emptyset$, we can minimize distance of $+q_n$ (similarly for $-q_n$) to $\overline{pq}$, and then use this to build a path transversal to $\overline{pq}$ with intersection number 1. Concatenating with l, we get a loop in $\tilde{\Sigma}$ whose intersection number with $\overline{pq}$ is equal to 1. But this loop is homotopic to the constant loop at e.g. $+q_n$ by simply-connectedness, a contradiction (see section 2.4 in [33]).

Now consider a minimizing geodesics $\gamma_n(t)$ connecting $\pm q_n$. First, $|\gamma_n| \to \infty$, since we can pick $\pm q_n$ to be in $\tilde{\Sigma} \setminus B_n(\overline{pq})$ for large n. We look at one of the intersection points of γ_n and $\overline{pq}$ named s_n and the velocity vector of γ_n at s_n named v_n, then $\{(s_n, v_n), n \in \mathbb{N}\}$ is contained in a compact set, and we have a subsequence converging to some point (s, v). The geodesic ray from s with velocity v (respectively $-v$) must have infinite length because we can get C^1_{loc}-convergence of these geodesics when $(s_n, v_n) \to (s, v)$. This means that we have found a geodesic line.

Now we can use the Toponogov's splitting theorem generalized to manifolds with convex boundary ([6, Theorem 5.2.2]) to conclude that $\tilde{\Sigma} = \mathbb{R} \times I$ where I is a connected one manifold with boundary, this is a contradiction to the boundary being strictly convex.

Step 4. We assume Σ is not compact and get a contradiction. In particular, we assume the diameter of $\partial\Sigma$ is unbounded. By the result of Step 3, we can assume that $\partial\Sigma$ is connected. The idea is that if the diameter of the boundary $\partial\Sigma$ is large, then we can prescribe a "capillary-minimizing" geodesic, and use stability inequality to get a contradiction.

The set up in Step 4 and Step 5 are written in general, not restricting to the ambient manifold being two dimensional. We use the notation $B_r(x)$ to denote a ball of radius r around x using the distance function on Σ, and $B'_r(x) \subset \partial\Sigma$ is a ball of radius r around $x \in \partial\Sigma$ using the distance function of $\partial\Sigma$ with respect to the induced metric.

Assume $w : \partial\Sigma \to [-1, 1]$ is the following Lipschitz function, for some fixed $x_0 \in \partial\Sigma$,

$$w(x) = \begin{cases} 1 & x \in B'_2(x_0) \\ \cos\rho(x) & -1 < w(x) < 1 \\ -1 & x \in \partial\Sigma \setminus B'_{2\pi}(x_0) \end{cases}$$

Here $\rho(y) : \partial\Sigma \to \mathbb{R}$ is a smooth function with$|\nabla\rho| < 1$. We denote $W_\pm := \{x \in \partial\Sigma : w(x) = \pm 1\}$ and require the set $\{x \in \partial\Sigma : \rho(x) = \pi\} = \partial W_-$ and the set $\{x \in \partial\Sigma : \rho(x) = 0\} = \partial W_+$ to be smooth submanifolds.

Now we minimize the functional in (3.2.2) among open sets with finite perimeter, containing the set $B'_2(x_0)$. Take such a Caccioppoli set Ω and let,

$$A(\Omega) = \mathcal{H}^1(\partial\Omega) - \int_{\Omega\cap\partial\Sigma} w, \tag{3.2.2}$$

where $\mathcal{H}^1(\partial\Omega)$ is the perimeter of $\partial\Omega$ in Σ.

We note that if a smooth minimizer $\partial\Omega \neq \emptyset$ exists, then for each component γ of $\partial\Omega$ with $\partial\gamma \neq \emptyset$, using the second variation formula [52], we have over the minimizing curve γ,

$$A''(\gamma) = \int_\gamma -\phi\Delta_\gamma\phi - (\mathrm{Ric}(N,N) + |\mathbb{I}_\gamma|^2)\phi^2 \\ + \int_{\partial\gamma} \phi\nabla_\nu\phi - \frac{\phi^2}{\sin^2\rho}\nabla_{\bar{\nu}}w + \frac{\phi^2}{\sin\rho}\langle\nabla_{\bar{\nu}}\bar{\nu}, \bar{N}\rangle - \phi^2\cot\rho\langle N, \nabla_\nu\nu\rangle,$$

where $\overline{N}$ is the outward unit normal of $\partial\Sigma \subset \Sigma$, N is the inward unit normal of $\partial\Omega \subset \Omega$, and $\overline{\nu}$ (respectively ν) is the outward unit normal of $\partial\Omega \cap \partial\Sigma \subset \Omega \cap \partial\Sigma$ (respectively $\partial\gamma \subset \gamma$).

We can plug in $\phi = 1$, use $\mathbb{I}_\gamma = 0 = \langle N, \nabla_\nu\nu\rangle$, and use the Gauss Equation,

$$0 \le A''(\gamma) = \int_\gamma -\frac{1}{2}R_\Sigma + \int_{\partial\gamma} \frac{1}{\sin\rho}(\langle\nabla_{\bar{\nu}}\bar{\nu}, \overline{N}\rangle + \nabla_{\bar{\nu}}\rho) \\ < \int_{\partial\gamma} \frac{1}{\sin\rho}(-1+1) = 0,$$

using $|\nabla\rho| < 1$ and $-k_{\partial\Sigma} = \langle\nabla_{\overline{\nu}}\overline{\nu}, \overline{N}\rangle \le -1$, leading to a contradiction.

Step 5. In this step we write down some technical details needed to show that a smooth minimizer exists (used in Step 4).

If Ω is a candidate in a minimizing sequence of A, by assumption we have $\partial\Omega \cap B'_2(x) = \emptyset$. In fact we can find an open neighborhood Ω' of $W_+ = \{x \in \partial\Sigma : w(x) = 1\}$ in Σ, so that any minimizing sequence must contain Ω'.

To elaborate, we assume without loss of generality each Ω in the minimizing sequence has smooth boundary, now we show that for some choice of Ω',

$$\delta(\Omega) := A(\Omega \cup \Omega') - A(\Omega) = \mathcal{H}^1(\partial\Omega' \setminus \Omega) - \mathcal{H}^1(\partial\Omega \cap \Omega') - \int_{\partial\Sigma\cap(\Omega'\setminus\Omega)} w \le 0.$$

We consider the following family $\Phi_t(x) := \exp\big(-\varphi_t(x)\overline{N}(x)\big)$ for $x \in \partial\Sigma$; recall $\overline{N}$ is the outward pointing unit normal of $\partial\Sigma \subset \Sigma$. Here $\varphi_t(x) := \max\{s\phi(x) + t, 0\}$ for some fixed small $s > 0$ to be chosen, and $\phi : \partial\Sigma \to [-1,1]$ is a smooth function, such that $\{x \in \partial\Sigma : \phi(x) > 0\} = W_+^\circ$, and $\nabla\phi(x) \ne 0$ for any $x \in \partial W_+$. Note that $\Gamma_t := \overline{\Phi_t(\partial\Sigma) \setminus \partial\Sigma}$ is a smooth submanifold in Σ, and as $s, t \to 0$, Γ_t converges to a smooth domain in $\partial\Sigma$. We denote the unit normal of Γ_t by ν_t, pointing in the direction as t increases. Then since $\nabla\phi(x) \neq 0$ for any $x \in \partial W_-$, we have $\nu_0(x) \cdot \overline{N}(x) > -1 = -w(x)$. We pick s' small enough, so that for any $t \in [-s, s']$ and $x \in \Gamma_t \cap \partial\Sigma$, $\nu_t(x) \cdot \overline{N}(x) \geq -w(x)$. Also because Γ_t converges to a smooth domain in $\partial\Sigma$ as $s, t \to 0$, using $k_{\partial\Sigma} \geq 1$, we know that $\mathrm{div}_{\Gamma_t}(\nu_t) \leq -0.5$.

Let $\Omega' := \cup_{t\in[-s,s']}\Gamma_t$ be the union of these "foliation", containing a tubular neighborhood of W_+. By divergence theorem for Lipschitz domains we have,

$$
\begin{aligned}
\delta(\Omega) &\leq \int_{\partial\Omega'\setminus\Omega} \nu_t \cdot \nu_{\partial\Omega'} - \int_{\partial\Omega\cap\Omega'} \nu_t \cdot \nu_{\partial\Omega} - \int_{\partial\Sigma\cap(\Omega'\setminus\Omega)} w \\
&= \int_{\Omega'\setminus\Omega} \mathrm{div}_{\Gamma_t}(\nu_t) + \int_{\partial\Sigma\cap(\Omega'\setminus\Omega)} \nu_t \cdot (-\overline{N}) - w \\
&\leq \int_{\Omega'\setminus\Omega} \mathrm{div}_{\Gamma_t}(\nu_t) \leq 0
\end{aligned}
$$

where we used $\nu_t \cdot \overline{N} \geq -w(x)$ for any $x \in \Gamma_t \cap \partial\Sigma$ and $\mathrm{div}_{\Gamma_t}(\nu_t) \leq -0.5$ for $t \in [-s, s']$. We get that any minimizing sequence must (eventually) contain Ω'.

We obtained for any minimizing sequence $\gamma_i = \partial\Omega_i$, we have $\Omega_i \supset \Omega' \supset W_+$ eventually. A similar argument shows that $\Omega_i \cap W_- = \emptyset$, so the term $\int_{\partial\Sigma\cap\Omega_i} w$ is uniformly bounded for a minimizing sequence, and $\mathcal{H}^1(\gamma_i)$ is also uniformly bounded. We get that the minimizing sequence must be contained in a bounded set from the point x_0. We can continue the minimization with standard BV compactness and regularity theory (see [17]).

We now check that the smooth minimizer $\gamma = \partial\Omega$ must have nonempty boundary. Note $B_2'(x_0) \subset \Omega' \subset \Omega$, and Ω is disjoint from the set W_-. So if γ is a minimizer with empty boundary, then $\Omega \cap \partial\Sigma$ and $\partial\Sigma \setminus \Omega$ is two disjoint nonempty open sets, whose union is $\partial\Sigma$, a contradiction to $\partial\Sigma$ being connected. □

3.3 Result for 3-Manifolds

Contrary to the 2-dimensional case, a 3-manifold with nonnegative scalar curvature and uniformly mean convex boundary, might not have connected boundary. We also might not have that any point is at bounded distance away from the boundary (using the same method would require non-negative Ricci curvature).

Before we start the proof of Theorem 3.1.4, we need the following lemma which is analogous to Lemma 16 in [13]. The inequality (3.3.1), when compared to the requirement of [13] in the PSC setting, is suitable for nonnegative scalar curvature. And we added the assumption (3.3.2), which is suitable for mean-convexity.

Lemma 3.3.1. *If there is a smooth function $u > 0$ over a compact surface $(\Sigma, \partial\Sigma)$ with nonempty boundary, such that,*

$$\Delta_\Sigma u \leq \frac{R_\Sigma}{2} u + \frac{|\nabla_\Sigma u|^2}{2u} \quad \textit{over } \Sigma, \tag{3.3.1}$$

$$\frac{\nabla_\nu u}{u} \geq a_0 - k_{\partial\Sigma} \quad \textit{over } \partial\Sigma, \tag{3.3.2}$$

then $d(x, \partial\Sigma) \leq \frac{2}{a_0}$ for any $x \in \Sigma$; the outward pointing unit normal along $\partial\Sigma$ is denoted as ν, R_Σ is the scalar curvature of Σ and $k_{\partial\Sigma}$ is the geodesic curvature of $\partial\Sigma$.

Proof. If not, assume $d(z, \partial\Sigma) \geq \frac{2}{a_0} + 2\delta$ for some $z \in \Sigma$ and $0 < \delta < 1$. We then find a minimizer of the following functional $\mathcal{F}(\Gamma)$ for sets with finite perimeters $\Gamma \subset \Sigma$ containing a neighborhood U_0 of z (will be chosen later) and disjoint from $\partial\Sigma$. Denote $\partial\Gamma = \gamma$, and ν_γ the outward unit normal of $\gamma \subset \Gamma$,

$$\mathcal{F}(\Gamma) = \int_\gamma u - \int_\Sigma hu(\chi_\Gamma - \chi_{\Gamma_0}),$$

here $h(y)$ is a mollification of the function $\hat{h}(y) = \frac{2}{\alpha - d_\Sigma(y, \partial\Sigma)} > 0$ when $y \in U$, with $\alpha = \frac{2}{a_0} + \delta$ and $U := \{y \in \Sigma, d(y, \partial\Sigma) < \alpha\}$. We require $h|_{\partial\Sigma} < a_0, h|_{\Sigma\setminus U} = \infty$, and $\frac{1}{2}h^2 - |\nabla h| > 0$ everywhere on U. We pick Γ_0 to be an open neighborhood of z with smooth boundary and $h|_{\Sigma\setminus\Gamma_0} \in L^\infty$. So for any smooth open set Γ, $\mathcal{F}(\Gamma) > -\infty$.

By the proof of Proposition 12 in [13], there is a smooth open neighborhood U_0 around z such that $\mathcal{F}(\Gamma \cup U_0) \leq \mathcal{F}(\Gamma)$ for any Γ with smooth boundary and $h|_{\Sigma\setminus U_0} \in L^\infty$. This implies that any minimizing sequence must contain U_0 and $\inf \mathcal{F} > -\infty$.

We now check that any minimizing sequence must be disjoint from a fixed open neighborhood of $\partial\Sigma$. So by interior regularity we have that a smooth minimizer exists

[61] [13].

By first variation, if a smooth minimizer Γ exists then its boundary γ has geodesic curvature equal to $k_\gamma = h - \frac{\nabla_\nu u}{u}$. If γ touches the boundary $\partial\Sigma$ at some point x, then,

$$\begin{aligned} k_\gamma(x) &= h|_{\partial\Sigma}(x) - \frac{\nabla_\nu u}{u}(x) \\ &\leq h|_{\partial\Sigma}(x) - a_0 + k_{\partial\Sigma}(x) < k_{\partial\Sigma}, \end{aligned} \tag{3.3.3}$$

where we used (3.3.2) and $h|_{\partial\Sigma} < a_0$. Using this observation, similar to Step 5 in Theorem 3.1.3, we use a foliation along $\partial\Sigma$ to modify our minimizing sequence so that it is disjoint from a fixed open neighborhood of $\partial\Sigma$.

To be precise, given any smooth minimizing sequence $\gamma_i = \partial\Gamma_i$, we show that $\mathcal{F}(\Gamma_i \setminus T') \leq \mathcal{F}(\Gamma_i)$, where $T' = \cup_{t\in[0,\epsilon],z\in\partial\Sigma} \exp_z(-t\nu)$ for some small ϵ to be decided. If we fix $t \in [0,\epsilon]$, then we get $T'_t = \cup_{z\in\partial\Sigma}\exp_z(-t\nu)$ a smooth curve with unit normal ν_t (pointing in the direction as t increases), and that as $t \to 0, \nu_t \to -\nu$. We denote the outward pointing unit normal of $\partial T' \subset T'$ and $\partial\Gamma_i \subset \Gamma_i$ as $\nu_{\partial T'}$ and $\nu_{\partial\Gamma_i}$. Then,

$$\begin{aligned} \mathcal{F}(\Gamma_i \setminus T') - \mathcal{F}(\Gamma_i) &= \int_{\partial T'\cap\Gamma_i} u + \int_{\partial\Gamma_i\cap T'} u\cdot(-1) + \int_{\Gamma_i\cap T'} hu \\ &\leq \int_{\partial T'\cap\Gamma_i} u(-\nu_t)\cdot\nu_{\partial T'} + \int_{\partial\Gamma_i\cap T'} u\nu_t\cdot\nu_{\partial\Gamma_i} + \int_{\Gamma_i\cap T'} hu \\ &= \int_{T'\cap\Gamma_i} \operatorname{div}^\Sigma(u\nu_t) + hu \\ &= \int_{T'\cap\Gamma_i} u\operatorname{div}^\Sigma(\nu_t) + \nabla_{\nu_t}u + hu \\ &= \int_{T'\cap\Gamma_i} u(\operatorname{div}^\Sigma(\nu_t) + \frac{\nabla_{\nu_t}u}{u} + h) \leq 0 \end{aligned}$$

in the final inequality we used (3.3.3). Indeed, as $t \to 0$, $\nu_t \to -\nu$, $\operatorname{div}(\nu_t) \to -k_{\partial\Sigma}$, so by (3.3.3),

$$h - \frac{\nabla_{-\nu_t}u}{u} + \operatorname{div}(\nu_t) < 0.$$

So any minimizing sequence must be disjoint from T' provided we choose ϵ small enough.

We now write out the second variation formula for a minimizer of $\mathcal{F}(\Gamma)$. This was derived in Theorem 6.3 of [65] (see also [13] Lemma 16),

$$
\begin{aligned}
0 &\leq \int_\gamma |\nabla_\gamma \psi|^2 u - \frac{1}{2} R_\Sigma \psi^2 u - \frac{1}{2} k_\gamma^2 \psi^2 u + (\Delta_\Sigma u - \Delta_\gamma u)\psi^2 - \langle \nabla_\Sigma u, \nu_\gamma \rangle \psi^2 h - \langle \nabla_\Sigma h, \nu_\gamma \rangle \psi^2 u \\
&\leq \int_\gamma |\nabla_\gamma \psi|^2 u - (\Delta_\gamma u)\psi^2 - (\frac{h^2}{2} + \nabla_{\nu_\gamma} h)\psi^2 u + \frac{\psi^2 (\nabla_\gamma u)^2}{2u} \\
&< \int_\gamma |\nabla_\gamma \psi|^2 u + \nabla_\gamma \psi^2 \nabla_\gamma u + \frac{\psi^2 (\nabla_\gamma u)^2}{2u} \quad (\star)
\end{aligned}
$$

where we used (3.3.1) in the second inequality and also $\frac{k_\gamma^2}{2} = \frac{h^2}{2} + \frac{(\nabla_{\nu_\gamma} u)^2}{2u} - \frac{h \nabla_{\nu_\gamma} u}{u}$. To get the strict inequality in $(\star)$ we used $\frac{1}{2} h^2 + \nabla_{\nu_\gamma} h > 0$. Then we can plug $\psi = u^{-\frac{1}{2}}$ into $(\star)$ to get $0 < \int_\gamma \frac{-1}{4u^2} (\nabla_\gamma u)^2$, a contradiction. So $d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}$ as claimed. $\square$

Remark 3.3.2. *One can check that in the above proof, it's sufficient to have equation (3.3.1): $\Delta_\Sigma u \leq \frac{R_\Sigma}{2} u + \frac{|\nabla_\Sigma u|^2}{2u}$ over the set $U := \{y \in \Sigma, d(y, \partial\Sigma) < \frac{2}{a_0} + 3\delta\}$ instead of requiring it everywhere in Σ, because (3.3.1) is only used in the step before $(\star)$ over the minimizer γ, which must lie in U by construction.*

Before proving Theorem 3.1.4, we start with the simple case which allows us to find capillary surfaces in a compact manifold with sufficiently large boundary diameter.

Lemma 3.3.3. *If $(V, \partial V, g)$ is a connected compact 3-manifold with $R_V \geq 0$, the mean curvature of the boundary satisfies $H_{\partial V} \geq \frac{\pi}{d_0} + a_0$ for some $a_0 > 0$, and ∂V is connected with intrinsic diam$(\partial V) > d_0 > 0$. Then there are finitely many capillary surfaces $(\Sigma_i)_{i=1}^k$ with nonempty boundary of bounded length: $|\partial \Sigma_i| \leq \frac{2\pi}{a_0}$, and $d_{\Sigma_i}(x, \partial \Sigma_i) \leq \frac{2}{a_0}$ for any $x \in \Sigma_i$. Furthermore, each Σ_i is a topological disc, and $\cup_{i=1}^k \Sigma_i$ separates ∂V.*

Proof. Let $\mathrm{diam}(\partial V) = d_{\partial V}(p, q) > d_0 + 5\delta$, for some $\delta > 0$ and $p, q \in \partial V$. Then we can build a smooth function $w : \partial V \to \mathbb{R}$ such that $w(x) = 1$ if $d_{\partial V}(p, x) \leq 2\delta$ and $w(x) = -1$ if $d_{\partial V}(p, x) \geq d_0 + 3\delta$, and when $-1 < w(x) < 1$, then $w(x) = \cos \rho(x)$ with $|\nabla^{\partial V} \rho| < \frac{\pi}{d_0}$.

We consider a minimizer of the following functional among open sets Ω with finite perimeters, containing $B^{\partial V}_{2\delta}(p)$ and disjoint from the set $\partial V \setminus B^{\partial V}_{d_0+3\delta}(p)$,

$$\mathcal{A}(\Omega) = |\partial\Omega| - \int_{\partial V\cap\Omega} w.$$

Claim: Any minimizing sequence Ω_i must (eventually) contain a fixed open neighborhood around $W_+ := \{x \in \partial V : w(x) = 1\}$ and (eventually) be disjoint from some fixed open neighborhood of $W_- := \{x \in \partial V : w(x) = -1\}$; the boundary of $\partial\Omega$, i.e. $\partial\Omega \cap \partial V$, of a minimizer Ω, must lie in the set $\{x \in \partial V : |w(x)| < 1\}$. We remark that the regularity for capillary surfaces (see [17],[11]) requires that $\partial\Omega \cap \partial V \subset \{x \in \partial V : |w(x)| < 1\}$.

The proof of these two claims is analogous to Step 5 of Theorem 3.1.3. Let $\overline{N}$ be the outward unit normal of $\partial V \subset V$, we consider the following family $\Phi_t(x) := \exp\big(-\varphi_t(x)\overline{N}(x)\big)$. Here $\varphi_t(x) := \max\{s\phi(x)+t, 0\}$ for some small $s > 0$ to be chosen, and $\phi : \partial V \to [-1, 1]$ is a smooth function, such that $\{x \in \partial V : \phi(x) > 0\} = W^\circ_+$, and $\nabla\phi(x) \neq 0$ for any $x \in \partial W_+$. The same argument as in Step 5 of Theorem 3.1.3 shows that each slice $\Sigma_t := \overline{\Phi_t(\partial V) \setminus \partial V}$ is a smooth surface with unit normal ν_t and mean curvature $H_t > 0$ and $\nu_t \cdot \overline{N}(x) \geq w(x)$ also holds for $x \in \Sigma_t \cap \partial V$ for small $t > 0$. We consider the foliation

$$\Omega' = \cup_{t\in[-s,s']}\Sigma_t$$

Then the same computation shows that,

$$\begin{aligned}
\mathcal{A}(\Omega' \cup \Omega) - \mathcal{A}(\Omega) &\leq \int_{\partial\Omega'\setminus\Omega} \nu_t \cdot \nu_{\partial\Omega'} - \int_{\partial\Omega\cap\Omega'} \nu_t \cdot \nu_{\partial\Omega} + \int_{\partial V\cap(\Omega'\setminus\Omega)} w \\
&= \int_{\Omega'\setminus\Omega} \mathrm{div}_{\Sigma_t}(\nu_t) + \int_{\partial V\cap(\Omega'\setminus\Omega)} \nu_t \cdot (-\overline{N}) + w \\
&\leq \int_{\Omega'\setminus\Omega} \mathrm{div}_{\Sigma_t}(\nu_t) \leq 0.
\end{aligned}$$

We finished the proof of the claim. So we find a minimizing Caccioppoli set Ω with smooth boundary $\Sigma = \partial\Omega \neq \emptyset$ since V is connected, and $\partial\Sigma \neq \emptyset$ since ∂V is connected. We note that $\partial\Omega$ might have many components, some of which are closed surface with no boundary. We want to examine each component Σ with non-empty boundary below.

First we want to apply Lemma 3.3.1 to get the distance bound. We write $\sqrt{1-w^2} = \sin\rho \neq 0$. We have the following first and second variation formulas over Σ (see also [52]); here we also used the Gauss-Codazzi equation $R_V = R_\Sigma + 2\operatorname{Ric}(N,N) + |\mathrm{II}_\Sigma|^2 - H_\Sigma^2$ in the second variation:

$$\begin{aligned} &H_\Sigma = 0, \quad \langle \nu, \overline{\nu} \rangle = \cos\rho; \\ &0 \le \int_\Sigma -\phi\Delta_\Sigma\phi - \frac{1}{2}(R_V - R_\Sigma + |\mathrm{II}_\Sigma|^2 + H_\Sigma^2)\phi^2 \\ &\quad + \int_{\partial\Sigma} \frac{\phi^2}{\sin\rho}(\langle \nabla_{\bar\nu}\bar\nu, \bar N\rangle + \nabla_{\bar\nu}\rho - \cos\rho\langle N, \nabla_\nu \nu\rangle) + \phi\nabla_\nu\phi. \end{aligned} \tag{3.3.4}$$

where $\overline{\nu}$ is the outward unit normal of $\partial\Sigma \subset \left(\overline{\Omega} \cap \partial V\right)$, ν is the outward unit normal of $\partial\Sigma \subset \Sigma$, N is the inward unit normal of $\Sigma \subset \overline{\Omega}$.

Now by (3.3.4) there is a smooth $u > 0$ over Σ with,

$$\Delta_\Sigma u + \frac{1}{2}(R_V - R_\Sigma + |\mathrm{II}_\Sigma| + H_\Sigma^2)u \le 0, \tag{3.3.5}$$

$$\nabla_\nu u + \frac{u}{\sin\rho}[\langle \nabla_{\bar\nu}\bar\nu, \bar N\rangle + \nabla_{\bar\nu}\rho - \cos\rho\langle \nabla_\nu\nu, N\rangle] = 0, \quad \text{along } \partial\Sigma \tag{3.3.6}$$

The following computation appeared in equation (3.8) in Li's paper [40] is very helpful, here γ' is a unit tangent vector along $\partial\Sigma$,

$$H_{\partial V} = -\langle \nabla_{\bar\nu}\bar\nu, \bar N\rangle + \cos\rho\langle N, \nabla_\nu\nu\rangle - \sin\rho\langle \nabla_{\gamma'}\gamma', \nu\rangle. \tag{3.3.7}$$

Combining equation (3.3.5),(3.3.6),(3.3.7) and $R_V \ge 0$, we have a smooth $u > 0$,

$$\begin{aligned} &\Delta_\Sigma u \le \frac{1}{2}R_\Sigma u \\ &\frac{\nabla_\nu u}{u} = \frac{1}{\sin\rho}(H_{\partial V} - \nabla_{\bar\nu}\rho) + \langle \nabla_{\gamma'}\gamma', \nu\rangle \ge \frac{\frac{\pi}{d_0} + a_0}{\sin\rho} - \frac{\pi}{d_0} - k_{\partial\Sigma} \ge a_0 - k_{\partial\Sigma}. \end{aligned} \tag{3.3.8}$$

So we can now apply Lemma 3.3.1 to get that $d_\Sigma(x, \partial\Sigma) \le \frac{2}{a_0}$ for all $x \in \Sigma$.

Now recall the Gauss Bonnet theorem (for a surface with nonempty boundary $\chi_\Sigma \leq 1$),

$$2\pi \geq 2\pi\chi_\Sigma = \int_\Sigma \frac{1}{2}R_\Sigma + \int_{\partial\Sigma} k_{\partial\Sigma} = \int_\Sigma \frac{1}{2}R_\Sigma - \int_{\partial\Sigma} \langle \nabla_{\gamma'}\gamma', \nu \rangle, \tag{3.3.9}$$

Combining (3.3.7), (3.3.9) and the second variation (3.3.4), we have that for $\phi = 1$,

$$\begin{aligned} 0 &\leq \int_\Sigma -\frac{1}{2}(R_V + |\mathrm{I\!I}_\Sigma|^2) + 2\pi\chi_\Sigma + \int_{\partial\Sigma} \frac{1}{\sin\rho}(\nabla_{\bar{\nu}}\rho - H_{\partial V}) \\ &\leq 2\pi - |\partial\Sigma| \cdot (-\frac{\pi}{d_0} + \frac{\pi}{d_0} + a_0) \\ &= 2\pi - |\partial\Sigma| \cdot a_0, \end{aligned} \tag{3.3.10}$$

note if $\chi_\Sigma \leq 0$, we would get a contradiction in (3.3.10). So any Σ with nonempty boundary, must be a topological disk. We now get $|\partial\Sigma| \leq \frac{2\pi}{a_0}$ as desired. ☐

We can now continue to the proof of Theorem 3.1.4, making suitable adaptions in the complete non-compact case.

Proof of Theorem 3.1.4. We first build one capillary surface $\Sigma = \Sigma_1$ using the same idea in Lemma 3.3.3 adapted to non-compact manifolds, then we build $\Sigma_2, \Sigma_3, \Sigma_4$... one by one.

We consider a minimizer of the following functional among open sets Ω with finite perimeter, containing $B_2^{\partial M}(x_0)$ (for some fixed $x_0 \in \partial M$) and contained in $B_{20}^M(x_0)$,

$$\mathcal{A}(\Omega) = |\partial\Omega| - \int_{\partial M \cap \Omega} w,$$

here $w : \partial M \to [-1, 1]$ is the following Lipschitz function,

$$w(x) = \begin{cases} 1 & x \in B_2^{\partial M}(x_0) \\ \cos\rho(x) & -1 < w(x) < 1 \\ -1 & x \in \partial M \setminus B_{2\pi}^{\partial M}(x_0) \end{cases}$$

Here $\rho(y) : \partial M \to \mathbb{R}$ is a smooth function with$|\nabla^{\partial M}\rho| \leq 1$. We denote $W_\pm :=$

$\{x \in \partial M : w(x) = \pm 1\}$ and require the set $\{x \in \partial M : \rho(x) = \pi\} = \partial W_-$ and the set $\{x \in \partial M : \rho(x) = 0\} = \partial W_+$ to be smooth submanifolds.

We want to apply the same proof of Lemma 3.3.3 to $B^M_{20}(x_0) \subset M$ so we need to perturb the metric of M near $\partial B^M_{20}(x_0)$ to get mean convexity. In particular, in a local coordinates near $\partial B^M_{20}(x_0)$ one can change the Christoffel symbols (which are first derivatives of the metric) while maintaining the coordinates being orthonormal (which is a condition only depending on the zero-th order derivatives of the metric). So we can perturb the metric near $\partial B^M_{20}(x_0)$ so that the mean curvature is at least 1.5, and the perturbation happens within $M \setminus B^M_{19}(x_0)$. We denote the scalar curvature after perturbation as $\hat{R}_M$, and $\hat{R}_M|_{B_{19}(x_0)} = R_M|_{B_{19}(x_0)}$.

Then we apply the same minimizing scheme as in Lemma 3.3.3, note we again have that the boundary of a minimizer must lie in $\{y \in \partial M, -1 < w(y) < 1\}$ and in this region $w(y)$ can be written as $w = \cos(\rho(y))$ for some smooth function ρ with $|\nabla^{\partial M}\rho| \leq 1$. So we find a smooth minimizer consisting of finitely many capillary surfaces, we now analyze each component Σ that has non-empty boundary.

Similar to (3.3.8) (here put $a_0 = 1, d_0 = \pi, H_{\partial M} \geq 2, |\nabla^{\partial M}\rho| \leq 1$), using the second variation, we have a smooth function $u : \Sigma \to (0, \infty)$,

$$\begin{aligned}
\Delta_\Sigma u &\leq \frac{1}{2}(R_\Sigma - \hat{R}_M)u \\
\frac{\nabla_\nu u}{u} &= \frac{1}{\sin\rho}(H_{\partial M} - \nabla_{\overline{\nu}}\rho) + \langle \nabla_{\gamma'}\gamma', \nu\rangle \geq 1 - k_{\partial\Sigma},
\end{aligned}$$

where $\overline{\nu}$ is the outward unit normal of $\partial\Sigma \subset \left(\overline{\Omega} \cap \partial M\right)$, ν is the outward unit normal of $\partial\Sigma \subset \Sigma$.

Using $\hat{R}_M|_{B_{19}(x_0)} = R_M|_{B_{19}(x_0)} \geq 0$ we have,

$$\Delta_\Sigma u \leq \frac{1}{2}(R_\Sigma - \hat{R}_M)u \leq \frac{1}{2}R_\Sigma u \quad \text{over the set } B_{19}(x_0).$$

Using Remark 3.3.2 to the set $U := \{y \in \Sigma, d(y, \partial\Sigma) < \frac{2}{1} + 3\} \subset B_{19}(x_0)$, we can now apply Lemma 3.3.1 to get that $d(x, \partial\Sigma) \leq 2$ for all $x \in \Sigma$. So $\Sigma \subset B_{19}(x_0)$, $\hat{R}_M|_\Sigma = R_M|_\Sigma$, and in the argument below we just write R_M.

Similar to (3.3.10) we have,

$$0 \leq \int_{\Sigma} -\frac{1}{2}(R_M + |\mathrm{II}_\Sigma|^2) + 2\pi\chi_\Sigma + \int_{\partial\Sigma} \frac{1}{\sin\rho}(\nabla_{\bar{\nu}}\rho - H_{\partial M}) \leq 2\pi - |\partial\Sigma|,$$

here again $\chi_\Sigma \leq 1$ since there is at least one boundary component; if $\chi_\Sigma \leq 0$, we would get a contradiction in (3.3.10). So any Σ with nonempty boundary, must be a topological disk, and we have $|\partial\Sigma| \leq 2\pi$ as desired.

In total, we have been able to construct finitely many capillary disks, we denote Σ_1 as the union of all components of $\partial\Omega$ that intersect the boundary ∂M, $Z_1 = \Omega \cap \partial M$, and each component Σ_1^α of Σ_1 have the following,

$$|\partial\Sigma_1^\alpha| \leq 2\pi, \quad d(y, \partial\Sigma_1^\alpha) \leq 2 \text{ for all } y \in \Sigma_1^\alpha,$$
$$B_2^{\partial M}(x) \subset Z_1, \quad \partial Z_1 = \partial\Sigma_1 \subset B_{2\pi}^{\partial M}(x) \setminus B_\pi^{\partial M}(x).$$

For the non-compact boundary ∂M, to get our desired exhaustion, we can replace the w function above by w_k, such that w_k is a mollification of $w_k = \cos\rho_k$ for ρ_k a mollification of $\tilde{\rho}_k = d_{\partial M}(x, \cdot) - k\pi$ on $B_{(k+1)\pi}^{\partial M}(x) \setminus B_{k\pi}^{\partial M}(x)$, and $|w_k| = 1$ elsewhere. For the corresponding $\mathcal{A}_k(\Omega)$, we can obtain a minimizer Ω_k, $\partial\Omega_k$ has finitely many components. We write $Z_k = \Omega_k \cap \partial M$. For each component Σ_k^α of Σ_k that intersects the boundary ∂M, we have

$$|\partial\Sigma_k^\alpha| \leq 2\pi, \quad d(y, \partial\Sigma_k^\alpha) \leq 2 \text{ for all } y \in \Sigma_k^\alpha,$$
$$B_{(k-1)\pi+2}^{\partial M}(x) \subset Z_k, \quad \partial Z_k = \partial\Sigma_k \subset B_{(k+1)\pi}^{\partial M}(x) \setminus B_{k\pi}^{\partial M}(x).$$

Since $B_{(k-1)\pi+2}^{\partial M}(x) \subset Z_k$, we have $\cup_k Z_k = \partial M$. For any $z \in \partial M \setminus Z_k$, any path from z to x must contain a point in $\partial\Sigma_k$ by connectedness of ∂M. We now obtained an exhaustion of ∂M via boundary of capillary surfaces of disk type, with length at most 2π.

If we know ∂M is simply connected, consider E_k an unbounded component of $\partial M \setminus Z_k$ such that $E_{k+1} \subset E_k$. Then as in [15] Proposition 3.2, for each k, ∂E_k must be connected, $\overline{E_k} \setminus E_{k+1}$ is also connected. We have $\sup_k \mathrm{diam}_{\partial M}(\overline{E_k} \setminus E_{k+1}) \leq 5\pi$.

Indeed, consider any two points $z_1, z_2 \in \overline{E_k} \setminus E_{k+1}$, then for each z_i there's some $y_i \in \partial\Sigma_k$, such that $d_{\partial M}(z_i, y_i) \leq 2\pi$. Now $|\partial\Sigma_k| \leq 2\pi$ implies $d_{\partial M}(y_1, y_2) \leq \pi$, adding up the length proves the claim. A similar argument also works if M is assumed to be simply connected instead of ∂M. □

Remark 3.3.4. *As a concrete example, take any surface Σ satisfies Theorem 3.1.3, i.e. with nonnegative scalar curvature and strictly convex boundary, then Theorem 3.1.4 applies to $\Sigma \times \mathbb{R}$. For example if Σ^2 is a strictly convex spherical cap, then Theorem 3.1.4 applies to any small perturbation of $\Sigma^2 \times \mathbb{R}$. We note that if M^3 has nonnegative Ricci curvature and strictly mean convex boundary, it might not split as $\Sigma \times \mathbb{R}$. For example consider capping off one end of a solid cylinder by a half-ball. If M^3 has nonnegative Ricci curvature, and strictly positive second fundamental form, then [41] has conjectured that it must be compact.*

Remark 3.3.5. *In the proof above the bound $d(x, \partial\Sigma) \leq 2$ for all $x \in \Sigma$ will depend on mean curvature of $\partial M \subset M$ and also on the angle function w prescribed on the boundary. In [52], the authors showed the disks and spherical caps are capillary stable surfaces of constant mean curvature in $\mathbb{B}^3$ when $w = \cos\theta$ is a constant function over $\partial\mathbb{B}^3$.*

Remark 3.3.6. *Theorem 3.1.4 only describes the behavior of ∂M, and has no control of the interior of M. As an example, if $B = \mathbb{B}^3_r(z)$ is a small interior ball in M^3 with $R_M|_B > 0$, then one can concatenate $M \setminus B$ along $\partial B \approx \mathbb{S}^2$ with $\mathbb{S}^2 \times [0, \infty)$ or to any complete (non-compact) manifold $(N, \partial N)$ with ∂N homeomorphic to $\mathbb{S}^2$, and $R_N|_{\partial N} > 0$. In particular, there is no equivalent of Corollary 3.1.5 for interior volume control even if one assumes M is simply connected. Endowing $\mathbb{S}^2 \times [0, \infty)$ with the spatial Schwarzschild metric as one interior end of M, the volume growth is Euclidean, instead of linear.*

We can now apply the same method to prove the bound for 1-Urysohn width if a surface can be filled in by a 3-manifold with a metric with strictly mean convex boundary and nonnegative scalar curvature.

Definition 3.3.7 (1-Urysohn width, [28],[34])**.** *Let X be a compact metric space, we say that X has Uryson 1-width bounded by L, if there is a graph G (a 1 dimensional* *simplecial complex) and a continuous map $f : X \to G$, such that every fiber $f^{-1}(y)$ for $y \in G$ has diameter bounded by L.*

Proof of Theorem 3.1.1. There is some $n \in \mathbb{N}$ so that $nd_0 < \mathrm{diam}(\partial N) \leq (n+1)d_0$. If $n \leq 1$ then we are done, otherwise let $d_{\partial N}(p,q) = \mathrm{diam}(\partial N) = nd_0 + 5\delta$, then we can obtain functions $(w_l)_{l=1}^n : \partial N \to \mathbb{R}$, $w_l(y) = 1$ if $d_{\partial N}(p,y) \leq 2\delta + (l-1)d_0$ and $w_l(y) = -1$ if $d_{\partial N}(p,y) \geq ld_0 + 3\delta$. When $-1 < w_l(y) < 1$ then $w_l = \cos \rho_l(x)$ for $|\nabla^{\partial N} \rho_l(y)| \leq \frac{\pi}{d_0}$ for any $y \in \partial N$.

We consider a minimizer of the following functional among open sets Ω with finite perimeters, containing $S_l := B^{\partial N}_{2\delta+(l-1)d_0}(p)$ and disjoint from the set $S_l' := \partial N \setminus B^{\partial N}_{ld_0+3\delta}(p)$,

$$\mathcal{A}(\Omega) = |\partial \Omega| - \int_{\partial N \cap \Omega} w_l.$$

Then we get (using the proof of Lemma 3.3.3) $\Sigma_l = \partial \Omega_l$ capillary surfaces with finitely many disk components $(\Sigma_l^k)_{k=1}^K$ and the boundary $\partial \Sigma_l^k$ of each component Σ_l^k must separate ∂N into two components using simply connectedness of ∂N. We also have $|\partial \Sigma_l^k| \leq \frac{2\pi}{a_0}$.

Now $\partial M \setminus (\cup_{l=1}^n \Sigma_l)$ is a union of finitely many 2-manifolds $\left(\Gamma_l^j\right)_{l=0}^n \subset \left(\overline{\Omega_{l+1}} \setminus \Omega_l\right)$, if $l = 0$ we think of Ω_l as the empty set, if $l = n$ we think of Ω_{l+1} as N. Each component Γ_l^j has piecewise smooth boundary, such that $\partial \Gamma_l^j \cap \partial \Sigma_l$ has at most one component (exactly 1 if $l \geq 1$) by simply connectedness. Indeed, if not then assume that there are $\partial \Sigma_l^1, \partial \Sigma_l^2$ two distinct boundary components in Γ_l^j, then take a fixed point $z \in \Gamma_l^j$ and distance-minimizing paths l_s to $\partial \Sigma_l^s$ for $s \in \{1,2\}$, concatenate l_1, l_2 together with a path in Ω_l transversal to both l_1, l_2 (for example by minimize distance to the point p), we get a contradiction in terms of intersection number with respect to each of l_1, l_2 using simply connectedness.

We show that each Γ_l^j has diameter bounded by $4d_0 + \frac{\pi}{a_0}$. If $l = 0$ or $l = n$ this is true by triangle inequality. We now look at $0 < l < n$, for any $z_1, z_2 \in \Gamma_l^j$, take the distance minimizing path to $\partial \Gamma_l^j \cap \partial \Sigma_l$ called t_1, t_2 respectively. The length of each t_1, t_2 is no more than $2d_0$ by construction. We have just shown that $\partial \Gamma_l^j \cap \partial \Sigma_l$ is a connected curve with length no more than $\frac{2\pi}{a_0}$, so the distance between any two points on the curve is no more than $\frac{\pi}{a_0}$. Adding these together we get the bound of diameter.

Using $\mathrm{diam}(\partial\Sigma_l^k) \leq \frac{\pi}{a_0}$ for each l, k, we can find some tubular neighborhood U_l^k of $\partial\Sigma_l^k$ in ∂M so that $\mathrm{diam}(U_l^k) \leq 4d_0 + \frac{\pi}{a_0}$.

We now define a graph G and a continuous map $l : \partial N \to G$. The graph G has vertices v_l^j and $l(x) = v_l^j$ if $x \in \Gamma_l^j \setminus \left(\cup_{lk} U_l^k\right)$. We connect two vertices v_l^j and $v_{l+1}^{j'}$ with an edge $E_{lj'} = [0,1]$ if Γ_l^j and $\Gamma_{l+1}^{j'}$ are separated by some $\partial\Sigma_{l+1}^k$. For a point z in the tubular neighborhood U_{l+1}^k homeomorphic to

$$\partial\Sigma_{l+1}^k \times [0,1] = \{(y,t), y \in \partial\Sigma_{l+1}^k, t \in [0,1]\},$$

we map by $l(z) = l(y,t) = t \in E_{lj'}$.

One can check that this gives us a continuous map of ∂N to a connected graph, the preimage of every point has diameter bounded by $4d_0 + \frac{\pi}{a_0}$. Using the definition of Urysohn width we have shown that the 1-Urysohn width of ∂N is bounded by $4d_0 + \frac{\pi}{a_0}$. □

Remark 3.3.8. *In Theorem 3.1.1 these quantities $R_g \geq 0, H_{\partial N} \geq \frac{\pi}{d_0} + a_0, U_1(\partial N) \leq 4d_0 + \frac{\pi}{a_0}$ scale accordingly. One can check that the minimum of*

$$(\frac{\pi}{d_0} + a_0)(4d_0 + \frac{\pi}{a_0}) = 5\pi + \frac{\pi^2}{a_0 d_0} + 4a_0 d_0,$$

is obtained when $a_0 d_0 = \frac{\pi}{2}$. So we can restate the theorem with $R_g \geq 0, H_{\partial N} \geq 3a_0$, and $U_1(\partial N) \leq 3\pi/a_0$.

Remark 3.3.9. *One can check that Theorem 3.1.1 also holds if X is complete non-compact, by using the same adaption for non-compact case in proof of Theorem 3.1.4. The proof of Theorem 3.1.1 also can be adapted if one assumes that M is simply connected instead of ∂M.*

3.4 Bandwidth Estimate

We now continue with the proof of the band width estimate.

Proof of Theorem 3.1.2. See Figure 1 (in introduction) as an example. We argue by contradiction. If $d_{\partial M}(\partial S_+, \partial S_-) > \pi + 2\delta$ for some $\delta > 0$, then there is a smooth function $w : \partial_0 M \to \mathbb{R}$, such that $|w(x)| = 1$ if $x \in \partial_0 M \setminus K$, $w(x) = \cos\rho(x)$ if $x \in K$, for a compact set $K \subset \partial_0 M$ with boundary $\partial_\pm K$ in $\partial_0 M$:

$$(K, \partial_\pm K) \Subset K_0 := \{x \in \partial_0 M, \delta < d_{\partial M}(x, \partial S_-) < \pi + 2\delta\},$$

and ρ is a smooth function with

$$\rho|_{\partial_- K} = \pi, \rho|_{\partial_+ K} = 0, |\nabla^{\partial M}\rho| < 1.$$

We may further assume $|w| = 1$ on $\partial M \setminus K$.

Now we minimize the following functional over open Caccioppoli sets Ω, such that $S_+ \subset \Omega, S_- \cap \Omega = \emptyset$:

$$\mathcal{A}(\Omega) = |\partial\Omega| - \int_{\partial M \cap \Omega} w. \tag{3.4.1}$$

Using $H_{\partial M} > 0$ and the same argument in section 3.3, we know any minimizing sequence has its boundary contained in the region $\{x \in \partial M, |\nabla w(x)| < 1\}$ (see also [60],[64]). By regularity of capillary problem [17] [11], we have a smooth minimizer $\Sigma' = \partial\Omega'$.

We also have $d_M(\Sigma, S_\pm) > 0$. So we have $S_- \subset \Omega'$, $\Omega' \cap S_+ = \emptyset$, $\partial\Omega' = \Sigma' \subset M_0$ and we can find a map of degree 1 (mod 2) from a component Σ of Σ' onto S_- as follows.

Consider a diffeomorphism $f : M_0 \to S_- \times [-1, 1]$, then for any $x = f^{-1}(z_0, t_0) \in \Sigma'$, we define the projection map $P(x) = z_0$. Consider a regular value $z \in S_-$ and the path $I_z := f^{-1}(z, t)$ in M_0 , we may write I_z just as (z, t) for $t \in [-1, 1]$. We look at the characteristic function of Ω', then $\chi_{\Omega'}(z, -1) = 1$ and $\chi_{\Omega'}(z, 1) = 0$, we note that z can be in ∂S_-, and $P^{-1}(\partial S_-) = \partial\Omega'$, so there is some component Σ of Σ' such that the degree of P is equal to 1 (mod 2). We also note that this means for any $x \in \partial S_-$, $P^{-1}(x) \cap \Sigma \neq \emptyset$. If ∂S_- has at least two components, then $\chi(\Sigma) \leq 0$, otherwise we have a map of degree 1 (mod 2) from $(\Sigma, \partial\Sigma) \to (S_-, \partial S_-)$.

Apply Kneser's theorem [36] to the double of Σ and S_-, we have $\chi(\Sigma) \leq 0$.

Therefore, using the stability inequality (see also 3.3.10) we get,

$$\begin{aligned}0 &\leq \int_{\Sigma} -\frac{1}{2}(R_M + |\mathbb{I}_{\Sigma}|^2) + \int_{\partial\Sigma} \frac{1}{\sin\rho}(\nabla_{\bar{\nu}}\rho - H_{\partial M}) + 2\pi\chi_{\Sigma} \\ &< \int_{\partial\Sigma} \frac{1}{\sin\rho}(1-1) + 2\pi\chi_{\Sigma} \leq 0\end{aligned}$$

a contradiction. □

Example 3.4.1. We give an example when M_0 is homeomorphic to $S_- \times [-1,1]$ with $\chi(S_-) > 0$, to show that in Theorem 3.1.2, the assumption of $\chi(S_-) \leq 0$ is crucial. Take the cylinder $\mathbb{D}^2 \times [-L, L] \subset \mathbb{R}^3$, with $\mathbb{D}^2$ unit disk in $\mathbb{R}^2$, and cap the top (or bottom) slice $\mathbb{D}^2 \times \{\pm L\}$ with an upper (or lower) hemisphere, then the length of the cylinder is unbounded when we let $L \to \infty$.

Remark 3.4.1. *We note that if in Theorem 3.1.2, we only assume $R_M \geq 0, H_0 \geq 1$, then by using the same argument of Theorem 3.1.4 (in a non-compact manifold, we can change the metric of M arbitrarily for points far away from the θ-bubble, using Remark 3.3.2), we can show that $d_M(S_+, S_-) \leq \pi + 4$. This weaker bound is obtained in the first version of the paper.*

Remark 3.4.2. *This bound is sharp and can be obtained by a very thin solid torus. Consider rotation the circle $\{(x,z), (x-R)^2 + z^2 = 1\}$ in the xz-plane along the z-axis in $\mathbb{R}^3$, then we get a solid torus with $R_M \geq 0$, $H_M \geq 1 - \epsilon(R)$, with $\epsilon(R) \to 0$ as $R \to \infty$. The top and the bottom of the torus has distance π.*

Chapter 4

Rigidity of Stable Free Boundary Minimal Hypersurfaces in $\mathbb{B}^4$

We prove that in the unit ball $\mathbb{B}^4$ of $\mathbb{R}^4$, there is no complete two-sided stable free boundary immersion. The result follows from a rigidity theorem of complete free boundary minimal hypersurfaces in complete 4-manifolds with non-negative intermediate Ricci curvature, convex boundary and weakly bounded geometry. The method uses warped θ-bubble, a generalization of capillary surfaces.

In section 2, we state some preliminaries and introduce the notations and set up of the paper. In section 3, we define "warped θ-bubbles" and derive its first and second variations. In section 4, we derive the inductive process and describe the inheritance phenomenon of warped θ-bubbles. In section 5, we study non-parabolic ends of free boundary minimal hypersurfaces in manifolds with non-negative intermediate Ricci curvature and 2-convex boundary conditions. In section 6, we prove the main results combining all the ingredients.

Results in this chapter comes from [67].

4.1 Introduction

We recall a complete two-sided stable minimal (free boundary) hypersurface $M^{n-1} \hookrightarrow X^n$ satisfies the following inequality, for any compactly supported test function ϕ,

$$\int_M |\nabla\phi|^2 - (\mathrm{Ric}_X(\nu_M, \nu_M) + |\mathrm{I\!I}_M|^2)\phi^2 - \int_{\partial M} \mathrm{I\!I}_{\partial X}(\nu_M, \nu_M)\phi^2 \geq 0, \tag{4.1.1}$$

for a choice of unit normal ν_M of $M \hookrightarrow X$, Ric_X the Ricci curvature of X and $\mathrm{I\!I}$ the corresponding second fundamental forms.

In the case that $\partial X \neq \emptyset$, Franz (Proposition 3.2.5 in [21]) obtained results for manifolds with "weakly positive geometry": If we assume $R_X \geq R_0 > 0$, $H_{\partial X} \geq 0$ and ∂X has no minimal components, or assume that $R_X \geq 0$, $H_{\partial X} \geq H_0 > 0$, then every complete two-sided stable free boundary minimal surface M, must be compact with intrinsic $\mathrm{diam}(M) \leq C(H_0, R_0)$, and is diffeomorphic to a disc. The result allows the author to obtain compactness results for compact embedded finite index minimal surfaces in a compact 3-manifold with weakly positive geometry, leading to a uniform bound of area, total curvature, genus and boundary components (Theorem 3.2.1 in [21]).

In higher dimensions, to obtain analogous rigidity or non-existence results, the assumption of $\mathrm{Ric} > 0$ or $R_g \geq 1$ needs to be strengthened ([15]). In this chapter, our main result shows that we can trade the uniformly positive scalar curvature assumption of X in [15] for uniformly positive mean curvature of ∂X.

Theorem 4.1.1. *Consider a 4-manifold $(X^4, \partial X)$ with weakly bounded geometry, assume $\mathrm{Ric}_2^X \geq 0, \mathrm{I\!I}_{\partial X} \geq 0$ and $H_{\partial X} \geq H_0 > 0$. If $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ is a complete two-sided stable free boundary minimal immersion, then M is totally geodesic and $\mathrm{Ric}_X(\nu_M, \nu_M) = 0$ along M, $\mathrm{I\!I}_{\partial X}(\nu_M, \nu_M) = 0$ along ∂M.*

In particular, a compact 4-manifold with non-negative sectional curvature and convex boundary, for example the unit ball in $\mathbb{R}^4$, satisfies our assumptions.

Corollary 4.1.2. *There is no complete two-sided stable free boundary minimal immersion in $\mathbb{B}^4$.*

To study PSC 3-manifolds, the μ-bubble method has been revisited to obtain prolific results including the Generalized Geroch Conjecture, the Stable Bernstein Theorem conjectured by Schoen and the Multiplicity One Conjecture of Marques and Neves ([29], [26], [69], [70],[68],[13], [14],[12]). The method of μ-bubble, as a generalization of minimal hypersurfaces, has proven to be a powerful tool to analyze the geometry of the ambient manifold.

In the case of manifolds with boundary, the analogy is given by capillary surfaces, which means constant mean curvature (CMC) surfaces having constant contact angle with the ambient manifolds' boundary. Li ([40]) used the method of capillary surfaces to show Gromov's dihedral rigidity conjecture for conical and prism type polyhedron. Chai and Wang ([9]) confirmed the conjecture for some cases of hyperbolic 3-manifold.

In chapter 3, we used capillary surfaces with prescribed (varying) contact angle, a notion called "θ-bubble", to study comparison results and geometry of manifolds with non-negative scalar curvature (NNSC) and uniformly mean convex boundary, obtaining sharp comparison results for surfaces, a 1-Urysohn width bound, a decomposition for the boundary of such manifolds, and a bandwidth estimate. Using capillary surfaces with prescribed contact angle, Ko and Yao ([37]) proved a smooth comparison and rigidity result analogous to Gromov's dihedral rigidity conjecture.

The method of θ-bubble in the above work for surfaces in 3-manifolds can be used inductively. We observe that in manifolds with NNSC and uniformly mean convex boundary, we have the PSC equivalent "inheritance"phenomenon. For a precise (and more general) statement, see Lemma 4.3.4.

Since we are now interested in hypersurfaces in a non-compact ambient manifold, it's important to control the number of ends of its stable minimal hypersurfaces. Cao, Shen and Zhu ([7]) proved that a complete stable minimal hypersurfaces in $\mathbb{R}^4$ has at most one end; a similar result on non-parabolic ends is obtained for manifolds with non-negative intermediate Ricci curvature in [15]. The case with FBMH $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ requires additional control of the boundary. We would like to use θ-bubbles (near the boundary) to exhuast the non-parabolic end. But a priori we don't know if the boundary ∂M is connected or disconnected, compact or non-compact, whether it's contained in the parabolic end or non-parabolic end.

Using Theorem Theorem 4.4.8, we can show if $(M^3, \partial M) \to (X^4, \partial X)$ is a FBMH and $\mathrm{Ric}_2^X \geq 0, \mathrm{II}_2^{\partial X} \geq 0$, assume M is non-parabolic and $\mathrm{Vol}(M) = \infty$, then each component of ∂M must be non-compact, and each component of the boundary ∂M must has an end in the only non-parabolic end of M.

We now provide a sketch of the proof of Theorem 4.1.1. We may pass to the universal cover and assume M is simply connected. The idea is similar to [15]. We want to control the volume growth of a stable FBMH M^3 in X^4 as of Theorem 4.1.1. The parabolic ends behave in a good way as one can find a harmonic function that approaches 1 everywhere while the Dirichlet energy goes to 0 when exhausting the parabolic end, which serves as a good test function to plug into the stability inequality (4.1.1). On a non-parabolic end, existence of a positive barrier function prevents us from using the same idea, so we want to exhaust a non-parabolic end with chunks with bounded diameter and volume, which shows that each non-parabolic end grows linearly in volume. This is achieved using θ-bubbles together with the control of the boundary ∂M and the non-parabolic end.

4.2 Definitions of θ-bubbles

We reserve the notation ∂M to denote the boundary of a manifold $(M, \partial M)$, and if Ω is an open subset of M, we denote the topological boundary as $\partial'\Omega$ and the closure of Ω as $\overline{\Omega}$. Then $\partial'\Omega \cap \partial M = \partial'(\Omega \cap \partial M)$ if Ω has Lipschitz boundary and $\overline{\Omega}$ intersect ∂M transversally.

In this section we introduce the notion of θ-bubble, a tool that is useful to probe the geometry of manifolds with non-negative scalar curvature (NNSC) and mean convex boundary, as an analogous notion to μ-bubble, first utilized by Gromov to probe the geometry of manifolds with positive scalar curvature (PSC).

We denote the reduced boundary of a Caccioppoli set Ω as $\partial^*\Omega$. Then if Ω is an open set with smooth boundary of a Riemannian manifold, $\partial^*\Omega = \partial'\Omega$.

Definition 4.2.1. Consider a Caccioppoli set Ω of a Riemannian manifold with boundary $(X^n, \partial X)$, a "θ-bubble" is a critical point to the following "prescribed contact angle" problem, among variations that send ∂X to itself,

$$\mathcal{A}_\theta(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega) - \int_{\partial X \cap \Omega} \cos\theta,$$

given a smooth function $\theta : \partial X \to \mathbb{R}$.

Capillary surfaces are surfaces of constant mean curvature and constant contact angle, and they are a critical point of $\mathcal{A}_\theta$ when θ is a constant function on ∂X and the volume ratio separated by the capillary surfaces is fixed. The θ-bubble can be thought as a "generalized capillary surfaces".

A more generalized notion called "warped θ-bubble" is adapted from the definition of θ-bubble with a weight on the ambient manifold, as an analogy to "warped μ-bubble".

Definition 4.2.2. Consider a Caccioppoli set Ω of a Riemannian manifold with boundary $(X^n, \partial X)$, a " warped θ-bubble" is a critical point to the following functional, among variations that send ∂X to itself,

$$\mathcal{A}_u(\Omega) = \int_{\partial^*\Omega} u d\mathcal{H}^{n-1} - \int_{\partial X \cap \Omega} u \cos\theta,$$

given a smooth function $\theta : \partial X \to \mathbb{R}$ and a positive smooth function $u : X \to \mathbb{R}_+$.

We first show that a minimizer exists under suitable assumptions and is smooth up to codimension 4 on the boundary.

Theorem 4.2.3. *If $(N^n, \partial N)$ is a compact connected Riemannian manifold with connected mean convex boundary $H_{\partial N} > 0$, let $\theta : \partial N \to \mathbb{R}$ be a smooth function such that $S_\pm = \{x \in \partial N, \cos\theta(x) = \pm 1\}$ and S_+ and S_- are open sets with smooth boundary in ∂N. Let $\mathcal{S}$ be the set of all Caccioppoli sets that contain S_+ and be disjoint from S_-, then*

$$\alpha = \inf_{\Omega \in \mathcal{S}} \mathcal{A}_\theta(\Omega)$$

is obtained by some $\Omega \in \mathcal{S}$. If $n \leq 4$, then the minimizer Ω is a smooth submanifold with boundary that intersect ∂N transversally.

Proof. We follow the arguments in [66] for surfaces and 3-manifolds.It's enough to show that there is a fixed open neighborhood Ω_+ of S_+ such that any minimizing sequence must contain Ω_+, and be disjoint from a fixed open neighborhood Ω_- of S_-. Then by connectedness of ∂N, a minimizer must exists and must intersect ∂N in a compact subset of $\{x \in \partial N, -1 < \cos\theta(x) < 1\}$ and the regularity results of [11] (see also [17]) applies, for $n \leq 4$ the minimizer is smooth and intersect ∂N transversally.

Consider the following family of foliations $\Phi_t(x) := \exp(-\varphi_t(x)\nu_{\partial N})$ where $\nu_{\partial N}$ is the outward pointing unit normal. Here $\varphi_t(x) := \{t_0\phi(x) + t, 0\}$ for some small fixed $t_0 > 0$ to be chosen, and $\phi : \partial N \to [-1, 1]$ is a smooth function such that $S_+ = \{x \in \partial N, \phi(x) \geq 0\}$ and $\nabla\phi(x) \neq 0$ for any $x \in \partial S_+$. Note that $\Gamma_t := \overline{\Phi_t(\partial N) \setminus \partial N}$ is a smooth submanifold in N, and as $t_0 \to 0, t \leq t_0$, Γ_t converge to S_+ smoothly, so $H_{\partial N} > 0$ implies $H_t := H_{\Gamma_t} > \epsilon$ for some $\epsilon > 0$.

Denote $\Omega_t := \cup_{-t_0 \leq s \leq t}\Gamma_t$ with outward pointing unit normal ν_{Γ_t} along the boundary Γ_t. Then we have that $\nu_{\partial N}(x) \cdot \nu_{\Gamma_0}(x) > -1 = -\cos\theta(x)$ for any $x \in \partial S_+$, which implies for small t_0 and $0 \leq t \leq t_0$, $\nu_{\partial N}(x) \cdot \nu_{\Gamma_t}(x) + \cos\theta(x) > 0$ for any $x \in \partial\Gamma_t$.

Then we take any candidate Ω (without loss of generality assume $\partial\Omega$ smooth) and compare,

$$\begin{aligned}
\mathcal{A}_\theta(\Omega_t \cup \Omega) - \mathcal{A}_\theta(\Omega) &= \mathcal{H}^{n-1}(\partial^*(\Omega_t \cup \Omega)) - \mathcal{H}^{n-1}(\partial^*\Omega) - \int_{\partial N \cap (\Omega_t \setminus \Omega)} \cos\theta \\
&\leq \int_{\partial^*\Omega_t \setminus \Omega} \nu_{\Gamma_t} \cdot \nu_{\Gamma_t} - \int_{\partial^*\Omega \cap \Omega_t} \nu_{\Gamma_t} \cdot \nu_{\partial\Omega} - \int_{\partial N \cap (\Omega_t \setminus \Omega)} \cos\theta \\
&= \int_{\Omega_t \setminus \Omega} \operatorname{div}(\nu_{\Gamma_t}) - \int_{\partial N \cap (\Omega_t \setminus \Omega)} \nu_{\Gamma_t} \cdot \nu_{\partial N} + \cos\theta \\
&\leq \int_{\Omega_t \setminus \Omega} -H_t \leq 0,
\end{aligned}$$

using when $t \leq t_0$ is small then $H_t > \epsilon > 0$; and the last inequality is sharp is $\Omega_t \setminus \Omega$ has nonzero measure.

A similar argument applies to show that any minimizer must be disjoint to some fixed neighborhood Ω_- of S_-. ☐

We can show that the same proof applies to the warped θ-bubbles provided the weight function u satisfies suitable boundary assumptions.

Theorem 4.2.4. *If $(N^n, \partial N)$ is a compact Riemannian manifold, let $\theta : \partial N \to \mathbb{R}$ be a smooth function such that $S_\pm = \{x \in \partial N, \cos\theta(x) = \pm 1\}$ and S_+ and S_- are open sets with smooth boundary in ∂N. Let $u > 0$ be a smooth function on N with $\nabla_{\nu_{\partial N}} u + uH_{\partial N} > 0$.*

Let $\mathcal{S}$ be the set of all Caccioppoli sets that contain S_+ and be disjoint from S_-, then

$$\alpha = \inf_{\Omega \in \mathcal{S}} \mathcal{A}_u(\Omega)$$

is obtained by some $\Omega \in \mathcal{S}$. If $n \leq 4$, then the minimizer Ω is a smooth submanifold with boundary that intersect ∂N transversally.

Proof. We apply exactly the same foliation as in Theorem 4.2.3, now we compare the following,

$$\begin{aligned}
\mathcal{A}_u(\Omega_t \cup \Omega) - \mathcal{A}_u(\Omega) &= \int_{\partial^*\Omega_t \backslash \Omega} u - \int_{\partial^*\Omega \cap \Omega_t} u - \int_{\partial N \cap (\Omega_t \backslash \Omega)} u \cos\theta \\
&\leq \int_{\partial^*\Omega_t \backslash \Omega} u\nu_{\Gamma_t} \cdot \nu_{\Gamma_t} - \int_{\partial^*\Omega \cap \Omega_t} u\nu_{\Gamma_t} \cdot \nu_{\partial\Omega} - \int_{\partial N \cap (\Omega_t \backslash \Omega)} u \cos\theta \\
&= \int_{\Omega_t \backslash \Omega} \operatorname{div}(u\nu_{\Gamma_t}) - \int_{\partial N \cap (\Omega_t \backslash \Omega)} u(\nu_{\Gamma_t} \cdot \nu_{\partial N} + \cos\theta) \\
&\leq \int_{\Omega_t \backslash \Omega} \operatorname{div}(u\nu_{\Gamma_t}) \\
&= \int_{\Omega_t \backslash \Omega} \nabla_{\nu_{\Gamma_t}} u + u \operatorname{div}_{\Gamma_t} \nu_{\Gamma_t}
\end{aligned}$$

Note as $t \to 0$,

$$\nabla_{\nu_{\Gamma_t}} u + u \operatorname{div}_{\Gamma_t} \nu_{\Gamma_t} \to -\nabla_{\nu_{\partial N}} u - uH_{\partial N} < 0.$$

Choosing t small enough then we are done. □

We compute the first and second variation of (warped) θ-bubbles.

Lemma 4.2.5 (First Variation). *If Ω is a smooth θ-bubble in $(N^n, \partial N)$ that intersect ∂N transversally, and $\partial^*\Omega = \Sigma$, then let ν be the outward pointing unit normal of $\partial\Sigma \subset \Sigma$ and $\overline{\nu}$ be the outward pointing unit normal of $\partial\Sigma \subset (\overline{\Omega} \cap \partial N)$, then*

$$H_\Sigma = 0, \langle \nu, \overline{\nu} \rangle = \cos\theta. \tag{4.2.1}$$

If Ω is a smooth warped θ-bubble in $(N^n, \partial N)$ that intersect ∂N transversally, and $\partial^*\Omega = \Sigma$ and a choice of unit normal ν_Σ, then

$$H_\Sigma = -\frac{\nabla_{\nu_\Sigma} u}{u}, \langle \nu, \overline{\nu} \rangle = \cos\theta. \tag{4.2.2}$$

Proof. The proof of (4.2.1) is the same as the case where θ is a constant function, as shown in [52]. We now prove (4.2.2). Let X_t be a vector field along Σ_t with $X_t(x) \in T_x\partial N$ for $x \in \partial N \cap \Sigma_t$ then,

$$\begin{aligned}
\frac{d}{dt}(\mathrm{Vol}(\Sigma_t)) &= \int_{\Sigma_t} \mathrm{div}_{\Sigma_t}(uX_t) - \int_{\partial\Sigma_t} u\cos\theta(X_t \cdot \overline{\nu}) \\
&= \int_{\Sigma_t} \nabla_{\Sigma_t} u \cdot X_t^\perp + u \,\mathrm{div}_{\Sigma_t} X_t^\perp + \int_{\partial\Sigma_t} uX_t \cdot \nu_t - u\cos\theta(X_t \cdot \overline{\nu}) \\
&= \int_{\Sigma_t} (\nabla_{\nu_{\Sigma_t}} u + H_{\Sigma_t} u) X_t \cdot \nu_{\Sigma_t} + \int_{\partial\Sigma_t} uX_t \cdot \overline{\nu_t}(\nu_t \cdot \overline{\nu_t} - \cos\theta),
\end{aligned}$$

where we used that for $t = 0$, $\nu_t = (X_t \cdot \overline{\nu_t})\overline{\nu_t} + (X_t \cdot \nu_{\partial N})\nu_{\partial N}$.

Setting the first variation equal to zero we obtain (4.2.2). ☐

Lemma 4.2.6 (Second Variation). *If Ω is a smooth warped θ-bubble in $(N^n, \partial N)$ that intersect ∂N transversally, denote $\partial^*\Omega = \Sigma$ and given a choice of unit normal ν_Σ, then let X_t be a vector field along Σ_t with $\Sigma_0 = \Sigma$ and $X_t(x) \in T_x\partial N$ for $x \in \partial N \cap \Sigma_t$, we denote $X_0 \cdot \nu_\Sigma =: \phi$, then*

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(\Sigma_t) = &\int_\Sigma -u\phi\Delta\phi - u\phi^2(\mathrm{Ric}_N(\nu_\Sigma, \nu_\Sigma) + |\mathrm{I\!I}_\Sigma|^2) + \phi^2(\Delta_N u - \Delta_\Sigma u) - \phi\langle \nabla^\Sigma\phi, \nabla^\Sigma u\rangle \\
&+ \int_{\partial\Sigma} u\phi\nabla_\nu\phi - \frac{u\phi^2}{\sin\theta}\left(\mathrm{I\!I}_{\partial N}(\overline{\nu}, \overline{\nu}) - \cos\theta\,\mathrm{I\!I}_\Sigma(\nu, \nu) - \nabla_{\overline{\nu}}\theta\right),
\end{aligned}$$

where ν is the outward pointing unit normal of $\partial\Sigma \subset \Sigma$ and $\overline{\nu}$ is the outward pointing unit normal of $\partial\Sigma \subset (\overline{\Omega} \cap \partial N)$.

Setting $u = 1$ we obtain the second variation of a θ-bubble.

A smooth (warped) θ-bubble is stable if for any admissible variation (if X_t is a vector field along Σ_t, then $\Sigma_0 = \Sigma$ and $X_t(x) \in T_x\partial N$ for $x \in \partial N \cap \Sigma_t$),

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(\Sigma_t) \geq 0.$$

Proof. We compute the integrand over Σ and first show it's enough to compute the second variation over $X_t^\perp = (X_t \cdot \nu_\Sigma)\nu_\Sigma$, let $X_t^T := X_t - X_t^\perp$,

$$\begin{aligned}
Q_1 :&= \int_\Sigma \frac{d}{dt}\bigg|_{t=0} (uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u) \\
&= \int_\Sigma \nabla_{X_t^T}(uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u) + \nabla_{X_t^\perp}(uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u) \\
&\overset{(*)}{=} \int_\Sigma \mathrm{div}_{\Sigma_t}((uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u) \cdot X_t^T) + \nabla_{X_t^\perp}(uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u) \\
&\overset{(*)}{=} \int_\Sigma \nabla_{X_t^\perp}(uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u),
\end{aligned}$$

where in the place where $(*)$ is marked we used the first variation $uH_\Sigma \cdot X_t + \nabla_{X_t^\perp} u = 0$.

We now compute Q_1 using $X_t = X_t^\perp = \phi_t \cdot \nu_t$ with $\nu_0 = \nu_\Sigma$, and evaluate at $t = 0$ at each step,

$$\begin{aligned}
Q_1 &= \int_\Sigma \partial_t(uH_\Sigma + \nabla_{\nu_\Sigma} u)\phi \\
&= \int_\Sigma \phi^2 \nabla_{\nu_\Sigma} u H_\Sigma + u\phi\partial_t H_\Sigma + \phi^2 \nabla_{\nu_\Sigma}\nabla_{\nu_\Sigma} u \\
&= \int_\Sigma -u\phi\Delta\phi - u\phi^2(\mathrm{Ric}_N(\nu_\Sigma, \nu_\Sigma) + |\mathrm{II}_\Sigma|^2) + \phi^2 \nabla_{\nu_\Sigma} u H_\Sigma + \phi^2(\nabla^2 u(\nu_\Sigma, \nu_\Sigma) + \nabla_{\nu_\Sigma}\nu_\Sigma \cdot \nabla^\Sigma u) \\
&\overset{(**)}{=} \int_\Sigma -u\phi\Delta\phi - u\phi^2(\mathrm{Ric}_N(\nu_\Sigma, \nu_\Sigma) + |\mathrm{II}_\Sigma|^2) + \phi^2(\Delta_N u - \Delta_\Sigma u) - \phi^2 \nabla^\Sigma \phi \cdot \nabla^\Sigma u
\end{aligned}$$

where in $(**)$ we used that $\Delta_N u - \Delta_\Sigma u = \nabla^2 u(\nu_\Sigma, \nu_\Sigma) + \nabla_{\nu_\Sigma} u H_\Sigma$ and $\phi\nabla_{\nu_\Sigma}\nu_\Sigma = -\nabla^\Sigma \phi$.

We now compute the integrand over $\partial\Sigma$. One can show that it's enough to consider variations $X_t = \varphi \cdot \overline{\nu}$ with $\phi = -\varphi \sin\theta$ (at $t = 0$) for this computation similar to the interior case,

$$\begin{aligned} Q_2 := & \int_{\partial\Sigma} u\varphi \frac{d}{dt}\bigg|_{t=0} (\nu_t \cdot \overline{\nu_t} - \cos\theta) \\ = & \int_{\partial\Sigma} u\varphi(\partial_t \nu_t \cdot \overline{\nu_t} + \nu_t \cdot \partial_t \overline{\nu_t} + \varphi \sin\theta \nabla_{\overline{\nu}}\theta) \\ = & \int_{\partial\Sigma} \frac{-u\phi}{\sin\theta}(\partial_t \nu_t \cdot \overline{\nu_t} + \nu_t \cdot \partial_t \overline{\nu_t} + \varphi \sin\theta \nabla_{\overline{\nu}}\theta) \end{aligned}$$

We compute $\partial_t \nu_t \cdot \overline{\nu_t}$ using $\overline{\nu_t} = \cos\theta \nu_t - \sin\theta_t \nu_{\Sigma_t}$. Note only at $t = 0$, $\theta_0 = \theta$ the prescribed function over ∂N,

$$\begin{aligned} \partial_t \nu_t \cdot \overline{\nu_t} &= \partial_t \nu_t \cdot (\cos\theta_t \nu_t - \sin\theta_t \nu_{\Sigma_t}) \\ &= -\sin\theta_t (\partial_t \nu_t \cdot \nu_{\Sigma_t}) \\ &= -\varphi \sin\theta_t (\cos\theta_t \nabla_{\nu_t} \nu_t \cdot \nu_{\Sigma_t} - \sin\theta_t \nabla_{\nu_{\Sigma_t}} \nu_t \cdot \nu_{\Sigma_t}) \\ &= -\phi \cos\theta \mathrm{II}_{\Sigma}(\nu, \nu) + \varphi \sin^2\theta (-\nu_t \cdot \nabla_{\nu_{\Sigma_t}} \nu_{\Sigma_t}) \\ &= -\phi \cos\theta \mathrm{II}_{\Sigma}(\nu, \nu) + \sin\theta(-\nu \cdot \nabla^{\Sigma}\phi) \quad \text{at } t = 0. \end{aligned}$$

And similarly,

$$\begin{aligned} \partial_t \overline{\nu_t} \cdot \nu_t &= \partial_t \overline{\nu_t} \cdot (\cos\theta_t \overline{\nu_t} + \sin\theta_t \nu_{\partial N}) \\ &= \sin\theta (\partial_t \overline{\nu_t} \cdot \nu_{\partial N}) \\ &= -\varphi \sin\theta \mathrm{II}_{\partial N}(\overline{\nu}, \overline{\nu}) \\ &= \phi \mathrm{II}_{\partial N}(\overline{\nu}, \overline{\nu}) \end{aligned}$$

In total we get,

$$Q_2 = \int_{\partial\Sigma} u\phi \nabla_{\nu}\phi - \frac{u\phi^2}{\sin\theta}(\mathrm{II}_{\partial N}(\overline{\nu}, \overline{\nu}) - \cos\theta \mathrm{II}_{\Sigma}(\nu, \nu) - \nabla_{\overline{\nu}}\theta)$$

This finishes the computation. ☐

Using Schoen-Yau rearrangement we can relate the interior terms in the second variation with scalar curvature. Using a computation in [40], we can rearrange the boundary terms to relate with the mean curvature of the ambient manifold.

Lemma 4.2.7 (Rearranged Second Variation)**.** *If Ω is a smooth warped θ-bubble in $(N^n, \partial N)$ that intersect ∂N transversally, denote $\partial^*\Omega = \Sigma$ and given a choice of unit normal ν_Σ, then let X_t be a vector field along Σ_t with $\Sigma_0 = \Sigma$ and $X_t(x) \in T_x\partial N$ for $x \in \partial N \cap \Sigma_t$, we denote $X_0 \cdot \nu_\Sigma =: \phi$, then*

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(\Sigma_t) = &\int_\Sigma - \mathrm{div}_\Sigma(u\nabla\phi)\phi - \frac{1}{2}u\phi^2(R_N - R_\Sigma + |\mathrm{II}_\Sigma|^2)\\ &+ \int_\Sigma -\frac{|\nabla_{\nu_\Sigma} u|^2\phi^2}{2u} + \phi^2(\Delta_N u - \Delta_\Sigma u)\\ &+ \int_{\partial\Sigma} u\phi\nabla_\nu\phi - \frac{u\phi^2}{\sin\theta}\left(H_{\partial N} - \cos\theta H_\Sigma - \nabla_{\overline{\nu}}\theta\right) + u\phi^2\mathrm{II}_\nu(\partial\Sigma)\end{aligned}$$

where ν is the outward pointing unit normal of $\partial\Sigma \subset \Sigma$, $\overline{\nu}$ is the outward pointing unit normal of $\partial\Sigma \subset (\overline{\Omega} \cap \partial N)$, and $\mathrm{II}_\nu(\partial\Sigma) = -\sum_i^{n-2}\langle\nabla_{e_i}e_i, \nu\rangle$ for an orthonormal basis (e_i) of $\partial\Sigma$.

Setting $u = 1$ we obtain the rearranged second variation of a θ-bubble.

Proof. We first prove,

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(\Sigma_t) = &\int_\Sigma -u\phi\Delta\phi - \frac{1}{2}u\phi^2(R_N - R_\Sigma + |\mathrm{II}_\Sigma|^2 + H_\Sigma^2)\\ &+ \int_\Sigma \phi^2(\Delta_N u - \Delta_\Sigma u) - \phi\langle\nabla^\Sigma\phi, \nabla^\Sigma u\rangle\\ &+ \int_{\partial\Sigma} u\phi\nabla_\nu\phi - \frac{u\phi^2}{\sin\theta}\left(H_{\partial N} - \cos\theta H_\Sigma - \nabla_{\overline{\nu}}\theta\right) + u\phi^2\mathrm{II}_\nu(\partial\Sigma)\end{aligned} \tag{4.2.3}$$

The interior rearrangement follows from the Gauss-Codazzi equation,

$$R_N = R_\Sigma + 2\,\mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) + |\mathrm{II}_\Sigma|^2 - H_\Sigma^2.$$

The boundary rearrangement follows from the following analogous computation of equation (3.8) in [40],

$$H_{\partial N} = \mathrm{II}_{\partial N}(\overline{\nu}, \overline{\nu}) + \cos\theta H_\Sigma - \cos\theta \mathrm{II}_\Sigma(\nu,\nu) + \sin\theta \mathrm{II}_\nu(\partial\Sigma).$$

This gives the equality (4.2.3).

The desired equality follows from first variation $uH_\Sigma + \nabla_{\nu_\Sigma} u = 0$. □

4.3 Inductive Process

We now start with the inductive process of proving diameter and circumference bounds for θ-bubbles. From this section onwards, any θ-bubble is assume to intersect with the ambient manifold's boundary transversally.

Lemma 4.3.1. *Let $(\Sigma^2, \partial\Sigma)$ be a compact surface with a positive smooth function $u : \Sigma \to \mathbb{R}_+$ such that,*

$$\Delta_\Sigma u \leq K_\Sigma u + \frac{|\nabla_\Sigma u|^2}{2u}, \quad \nabla_\nu u \geq a_0 u - \kappa_{\partial\Sigma} u,$$

for some $a_0 > 0$, where K_Σ is the Gauss curvature of Σ and $k_{\partial\Sigma}$ the geodesic curvature (equal to the mean curvature), and ν the outward pointing unit normal of $\partial\Sigma \subset \Sigma$; then for any $x \in \Sigma$,

$$d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}.$$

Proof. We summarize the proof in [66] and the reader can refer to [66] for a more detailed proof.

Assume that there is a point $p \in \Sigma$ with $d_\Sigma(p, \partial\Sigma) = \frac{2}{a_0} + 2\epsilon$ for some $\epsilon > 0$. Let

$$h(x) = \frac{2}{\frac{2}{a_0} + \epsilon - d_\Sigma(x, \partial\Sigma)}, \quad \text{if } d_\Sigma(x, \partial\Sigma) < \frac{2}{a_0} + \epsilon,$$

where we denote $d_\Sigma(x, \partial\Sigma)$ the mollified distance function to the boundary and that $h^2 - 2|\nabla^\Sigma h| > 0$. Let Ω_0 be a fixed open set containing $\{x \in \Sigma, d_\Sigma(x, \partial\Sigma) \geq \frac{2}{a_0} + \epsilon\}$.

We solve for the following warped μ-bubble problem over Σ for Caccioppolli sets in Σ containing a fixed open neighborhood of p, and be disjoint from a fixed open neighborhood of $\partial\Sigma$,

$$\mathcal{A}_h(\Omega) = \int_{\partial\Omega} u - \int_\Sigma hu(\chi_\Omega - \chi_{\Omega_0}).$$

A maximum principle argument of first variation in [66] shows that a smooth minimizer must exists. This is where we need the boundary mean convexity assumption $\partial_\nu u \geq a_0 u - \kappa_{\partial\Sigma} u$.

Then the first variation of the minimizer $\Gamma = \partial\Omega$, along a smooth variation $X = \phi\nu_\Gamma$ is the following,

$$\frac{d}{dt}\mathcal{A}_h(\Gamma) = \int_\Gamma \phi(uH_\Gamma + \nabla_{\nu_\Gamma} u) - hu\phi. \tag{4.3.1}$$

And the second variation is,

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{A}_h(\Gamma) &= \int_\Gamma \phi u(-\Delta_\Gamma \phi - \phi(\mathrm{Ric}_\Sigma(\nu,\nu) + |\mathrm{II}_\Gamma|^2)) + \phi^2 H_\Sigma \nabla_{\nu_\Gamma} u \\
&+ \int_\Gamma \phi^2 \nabla_{\nu_\Gamma}\nabla_{\nu_\Gamma} u - \phi^2 \nabla_{\nu_\Gamma}(hu) \\
&\overset{(\star 1)}{=} \int_\Gamma \phi u(-\Delta_\Gamma \phi - \phi(\frac{1}{2}R_\Sigma + |\mathrm{II}_\Gamma|^2)) + \phi^2 H_\Sigma \nabla_{\nu_\Gamma} u \\
&+ \int_\Gamma \phi^2 \nabla^2 u(\nu_\Gamma, \nu_\Gamma) + \phi^2 \nabla^\Gamma u \cdot \nabla_{\nu_\Gamma}\nu_\Gamma - \phi^2 \nabla_{\nu_\Gamma}(hu) \\
&\overset{(\star 2)}{=} \int_\Gamma \phi(-\operatorname{div}_\Gamma(u\nabla_\Gamma \phi) - u\phi(\frac{1}{2}R_\Sigma + |\mathrm{II}_\Gamma|^2)) + \phi^2(\Delta_\Sigma u - \Delta_\Gamma u) - \phi^2 \nabla_{\nu_\Gamma}(hu) \\
&\overset{(\star 3)}{\leq} \int_\Gamma \phi(-\operatorname{div}_\Gamma(u\nabla_\Gamma \phi)) + \frac{|\nabla_\Sigma u|^2}{2u}\phi^2 - u\phi^2|\mathrm{II}_\Gamma|^2 - \phi^2\Delta_\Gamma u - \phi^2 \nabla_{\nu_\Gamma}(hu) \\
&\overset{(\star 4)}{\leq} \int_\Gamma \phi(-\operatorname{div}_\Gamma(u\nabla_\Gamma \phi)) + \frac{|\nabla_\Sigma u|^2}{2u}\phi^2 - \frac{u\phi^2}{2}(h - \frac{\nabla_{\nu_\Gamma} u}{u})^2 - \phi^2\Delta_\Gamma u - \phi^2 \nabla_{\nu_\Gamma}(hu) \\
&= \int_\Gamma u(\nabla_\Gamma \phi)^2 + \frac{|\nabla_\Sigma u|^2}{2u}\phi^2 - \frac{u\phi h^2}{2} - \frac{(\nabla_{\nu_\Gamma} u)^2}{2u} - \phi^2 \Delta_\Gamma u - \phi^2 u \nabla_{\nu_\Gamma} h \\
&= \int_\Gamma \frac{|\nabla_\Gamma u|^2}{2u}\phi^2 - \phi^2 \Delta_\Gamma u - \frac{\phi^2 u}{2}(h^2 + 2\nabla_{\nu_\Gamma} h) + u(\nabla_\Gamma \phi)^2 \\
&\overset{(\star 5)}{<} \int_\Gamma \frac{|\nabla_\Gamma u|^2}{2u}\phi^2 - \phi^2 \Delta_\Gamma u + u(\nabla_\Gamma \phi)^2
\end{aligned}$$

$$= \int_\Gamma \frac{|\nabla_\Gamma u|^2}{2u}\phi^2 + \nabla_\Gamma u \nabla_\Gamma \phi^2 + u(\nabla_\Gamma \phi)^2.$$

The computation of warped μ-bubble here works in general dimensions with small adaptions. In ($\star$1) we used for surfaces the Ricci curvature is one half of the scalar curvature; in general dimensions one can use the Gauss-Codazzi equation. In ($\star$2) we used that for a normal variation with speed ϕ, $\partial_t \nu_\Gamma = -\nabla^\Gamma \phi$. In ($\star$3) we used the interior NNSC assumption $\Delta_\Sigma u \leq K_\Sigma u + \frac{|\nabla_\Sigma u|^2}{2u}$. In ($\star$4) we used the first variation $|\mathrm{I\!I}_\Gamma|^2 = (H_\Gamma)^2 = (h - \frac{\nabla_{\nu_\Gamma} u}{u})^2$ and $|\mathrm{I\!I}_\Gamma|^2 \geq \frac{|\mathrm{I\!I}_\Gamma|^2}{2}$. Lastly in ($\star$5) we used $h^2 - 2|\nabla^\Sigma h| > 0$.

We now choose $\phi^2 u = 1$ to get,

$$\frac{|\nabla_\Gamma u|^2}{2u}\phi^2 + \nabla_\Gamma u \nabla_\Gamma \phi^2 + u(\nabla_\Gamma \phi)^2 = \frac{|\nabla_\Gamma u|^2}{2u^2} - \frac{|\nabla_\Gamma u|^2}{u^2} + \frac{|\nabla_\Gamma u|^2}{4u^2} = -\frac{|\nabla_\Gamma u|^2}{4u^2},$$

which gives us a contradiction to the stability inequality. □

Lemma 4.3.2. *Let $(\Sigma^2, \partial\Sigma)$ be a stable u-warped θ-bubble in a 3-manifold $(M^3, \partial M)$ with,*

$$\Delta_M u \leq \frac{R_M}{2} u + \frac{|\nabla^M u|^2}{2u},$$

and over the boundary $\partial\Sigma$ we have,

$$\nabla_{\nu_{\partial M}} u + u(H_{\partial M} - \nabla_{\bar{\nu}}\theta) \geq (\sin\theta) a_0 u > 0 \tag{4.3.2}$$

then we have $|\partial\Sigma| \leq \frac{2\pi}{a_0}$ and $d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}$ for all $x \in \Sigma$.

Proof. The rearranged second variation over Σ gives,

$$\begin{aligned} 0 \leq & \int_\Sigma -\mathrm{div}_\Sigma(u\nabla\phi)\phi - \frac{1}{2}u\phi^2(R_M - R_\Sigma + |\mathrm{I\!I}_\Sigma|^2) \\ & + \int_\Sigma -\frac{|\nabla_{\nu_\Sigma} u|^2\phi^2}{2u} + \phi^2(\Delta_M u - \Delta_\Sigma u) \\ & + \int_{\partial\Sigma} u\phi\nabla_\nu\phi - \frac{u\phi^2}{\sin\theta}(H_{\partial M} - \cos\theta H_\Sigma - \nabla_{\bar{\nu}}\theta) + u\phi^2 \mathrm{I\!I}_\nu(\partial\Sigma) \\ \leq & \int_\Sigma -\mathrm{div}_\Sigma(u\nabla\phi)\phi + \frac{1}{2}u\phi^2(R_\Sigma - |\mathrm{I\!I}_\Sigma|^2) + \frac{|\nabla^\Sigma u|^2\phi^2}{2u} + \phi^2(-\Delta_\Sigma u) \end{aligned}$$

$$+\int_{\partial\Sigma} u\phi\nabla_\nu\phi - \frac{u\phi^2}{\sin\theta}\left(H_{\partial M} - \cos\theta H_\Sigma - \nabla_{\bar{\nu}}\theta\right) + u\phi^2 \mathrm{II}_\nu(\partial\Sigma)$$

Then we obtain that for some choice of $\phi > 0$,

$$\begin{aligned}
\operatorname{div}_\Sigma(u\nabla\phi) &\leq \frac{R_\Sigma}{2}u\phi - \phi\Delta_\Sigma u + \frac{|\nabla^\Sigma u|^2\phi}{2u}, \\
\nabla_\nu\phi &= \frac{\phi}{\sin\theta}(H_{\partial M} - \nabla_{\bar{\nu}}\phi - \cos\theta H_\Sigma) - \phi\mathrm{II}_\nu(\partial\Sigma)
\end{aligned}$$

Let $f = u\phi$, then

$$\begin{aligned}
\Delta_\Sigma f &= \operatorname{div}_\Sigma(u\nabla_\Sigma\phi) + \phi\Delta_\Sigma u + \nabla^\Sigma u \cdot \nabla^\Sigma\phi \\
&\leq \frac{R_\Sigma}{2}u\phi + \nabla^\Sigma u \cdot \nabla^\Sigma\phi + \frac{|\nabla^\Sigma u|^2\phi}{2u} \\
&\leq \frac{R_\Sigma}{2}u\phi + \frac{|\nabla_\Sigma(u\phi)|^2}{2u\phi} = \frac{R_\Sigma f}{2} + \frac{|\nabla_\Sigma f|^2}{2f}.
\end{aligned}$$

We now check the boundary condition of f. We first claim that for ν the outward pointing unit normal of $\partial\Sigma \subset \Sigma$

$$\nabla_\nu u + \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar{\nu}}\theta - \cos\theta H_\Sigma) \geq a_0 u > 0 \tag{4.3.3}$$

Using (4.3.3) we get,

$$\begin{aligned}
\partial_\nu f &= \phi\partial_\nu u + u\partial_\nu\phi \\
&= \phi\partial_\nu u + \frac{u\phi}{\sin\theta}(H_{\partial M} - \nabla_{\bar{\nu}}\theta - \cos\theta H_\Sigma) - u\phi\mathrm{II}_\nu(\partial\Sigma) \\
&\geq a_0(u\phi) - \mathrm{II}_\nu(\partial\Sigma)(u\phi) = a_0 f - \mathrm{II}_\nu(\partial\Sigma)f
\end{aligned}$$

This implies that $d_\Sigma(x,\partial\Sigma) \leq \frac{2}{a_0}$ for all $x \in \Sigma$ using Lemma 4.3.1.

The proof of (4.3.3) uses the relationship of normal and conormal vectors of a θ-bubble.

$$\begin{aligned}
\nu := \nu_{\partial\Sigma} &= \cos\theta\bar{\nu} + \sin\theta\nu_{\partial M}, & \nu_\Sigma &= -\sin\theta\bar{\nu} + \cos\theta\nu_{\partial M} \\
\nabla_{\nu_\Sigma} u &= -\sin\theta\nabla_{\bar{\nu}}u + \cos\theta\nabla_{\nu_{\partial M}}u, & \nabla_\nu u &= \cos\theta\nabla_{\bar{\nu}}u + \sin\theta\nabla_{\nu_{\partial M}}u
\end{aligned}$$

This implies

$$\begin{aligned}
&\nabla_\nu u + \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar\nu}\theta - \cos\theta H_\Sigma)\\
&= \nabla_\nu u - \frac{\cos\theta}{\sin\theta}(-\nabla_{\nu_\Sigma} u) + \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar\nu}\theta)\\
&= \cos\theta\nabla_{\bar\nu} u + \sin\theta\nabla_{\nu_{\partial M}} u - \cos\theta\nabla_{\bar\nu} u + \frac{\cos^2\theta}{\sin\theta}\nabla_{\nu_{\partial M}} u + \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar\nu}\theta)\\
&= \frac{1}{\sin\theta}\nabla_{\nu_{\partial M}} u + \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar\nu}\theta) \geq a_0 u > 0.
\end{aligned}$$

Finally, we can use Gauss-Bonnet in the second variation formula. When choosing $u\phi^2 = 1$, one can simplify to get,

$$\begin{aligned}
0 &\leq \int_\Sigma u(\nabla^\Sigma u^{-\frac{1}{2}})^2 + \frac{1}{2}R_\Sigma + \frac{|\nabla^\Sigma u|^2}{2u^2} - \frac{\Delta_\Sigma u}{u} + \int_{\partial\Sigma} -\frac{1}{\sin\theta}(H_{\partial M - \cos\theta H_\Sigma} - \nabla_{\bar\nu}\theta) + k_{\partial\Sigma}\\
&= \int_\Sigma -\frac{1}{4}u^{-1}|\nabla^\Sigma u|^2 + \int_\Sigma \frac{R_\Sigma}{2} + \int_{\partial\Sigma} k_{\partial\Sigma} + \int_{\partial\Sigma} -\frac{\nabla_\nu u}{u} - \frac{1}{\sin\theta}\left(H_{\partial M} - \cos\theta H_\Sigma - \nabla_{\bar\nu}\theta\right)\\
&\leq \int_\Sigma -\frac{1}{4}u^{-1}|\nabla^\Sigma u|^2 + 2\pi\chi_\Sigma - \int_{\partial\Sigma} a_0,
\end{aligned}$$

Since χ_Σ is a θ-bubble ($\partial\Sigma \neq \emptyset$), $\chi_\Sigma \leq 1$ implies $|\partial\Sigma| \leq \frac{2\pi}{a_0}$. □

Corollary 4.3.3. *Consider* $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ *a FMBH and* X *has*

$$R_X \geq 0, \quad H_{\partial X} \geq H_0 > 0.$$

Let the diameter of ∂M *(with respect to* $d_{\partial M}(\cdot,\cdot)$ *under the induced metric) be larger than some* $d_0 > 0$*, and assume*

$$H_0 - \frac{\pi}{d_0} = a_0 > 0,$$

then we can find a stable warped θ*-bubble* $(\Sigma^2, \partial\Sigma)$ *such that,*

$$|\partial\Sigma| \leq \frac{2\pi}{a_0}, \quad d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}.$$

Proof. Using the stability inequality of $(M, \partial M)$, we can find a positive $u : M \to \mathbb{R}_+$

$$\Delta_M u \leq \frac{1}{2} R_M, \quad \nabla_{\nu_{\partial M}} u = \mathrm{II}_{\partial X}(\nu_M, \nu_M) u.$$

Now using the diameter of ∂M is at least d_0, we want to use the existence of smooth minimizer in Theorem 4.2.4, for $S_\pm = \{x \in \partial M, \cos\theta(x) = \pm 1\}$, and we can assume that $|\nabla^{\partial M} \theta| \leq \frac{\pi}{d_0}$. We check that in this case,

$$\nabla_{\nu_{\partial M}} u + u H_{\partial M} = u H_{\partial X} \geq u H_0 > 0.$$

We now minimize the following functional as in Theorem 4.2.4 to get a stable u-warped θ-bubble,

$$\mathcal{A}_u(\Omega) = \int_{\partial^* \Omega} u - \int_{\partial M \cap \Omega} u \cos\theta.$$

We remark that now the ambient manifold M is not compact (in Theorem 4.2.4 we required compactness). We can use a similar adaptation to non-compact manifolds, as in the proof of Theorem 1.4 and Remark 3.2 in [66]. This is because the condition $H_0 - \frac{\pi}{d_0}$ will constrain the minimizer $\Sigma = \partial\Omega$ to lie close to its boundary $\partial\Sigma$, which was prescribed to lie in a bounded region. We refer the reader to [66] for more details.

We now check that condition (4.3.2) is satisfied.

$$\begin{aligned}\nabla_{\nu_{\partial M}} u + u(H_{\partial M} - \nabla_{\bar{\nu}} \theta) &= \mathrm{II}_{\partial X}(\nu_M, \nu_M) u + u(H_{\partial M} - \nabla_{\bar{\nu}} \theta) \\ &= u(H_{\partial X} - \nabla_{\bar{\nu}} \theta) \geq u(H_0 - \frac{\pi}{d_0}) = a_0 u \geq (\sin\theta) a_0 u > 0\end{aligned}$$

Now Lemma 4.3.2 applies to give the desired bounds for the θ-bubble. □

Finally we summarize the inductive procedure for smooth stable warped θ-bubble in a manifold of general dimension with an analogous NNSC and strictly mean convexity assumption. The following lemma can be viewed as the PSC equivalent "inheritance" phenomenon for manifolds with NNSC and mean convex boundary. The proof is the same as Lemma 4.3.2.

Lemma 4.3.4. *If* $(\Sigma^n, \partial\Sigma) \hookrightarrow (M^{n+1}, \partial M)$ *is a smooth stable* u*-warped* θ*-bubble,* *with $u > 0$ a smooth function on M, and for some $a_0 > 0$,*

$$\Delta_M u \leq \frac{R_M}{2} u + \frac{|\nabla^M u|^2}{2u}$$
$$\nabla_{\nu_{\partial M}} u \geq (\sin\theta) a_0 u - \frac{u}{\sin\theta}(H_{\partial M} - \nabla_{\bar{\nu}}\theta)$$

then $(\Sigma^n, \partial\Sigma)$ has for some $f > 0$,

$$\Delta_\Sigma f \leq \frac{R_\Sigma}{2} f + \frac{|\nabla^\Sigma f|^2}{2f}$$
$$\nabla_{\nu_{\partial\Sigma}} f \geq a_0 f - \mathrm{II}_\nu(\partial\Sigma) f$$

4.4 Parabolicity and Stability

In [7], the authors proved that a complete non-compact two-sided stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 3$ must have only one end. The use of Michael-Simon-Sobolev inequality for minimal hypersurfaces in $\mathbb{R}^{n+1}$ is crucial since this implies that any end of a stable minimal hypersurfaces must be non-parabolic. And having two non-parabolic ends together with the Schoen-Yau rearranged stability inequality using the Bochner formula, this implies that the stable minimal hypersurface must admit a non-constant harmonic function with finite energy and must have finite volume, a contradiction to the monotonicity formula.

In [15], the authors proved that under the assumption of non-negative 2 intermediate Ricci curvature of a 4 dimensional ambient manifold, a stable minimal hypersurface with infinite volume can have at most one non-parabolic end. This was extended to the FBMH case in chapter 2.

In section 4.5, we will show that if M is a simply connected stable minimal hypersurface in $(X^4, \partial X)$ as in our main theorem, then ∂M must be connected and have an end in the only non-parabolic end of M. In this section we prove some auxiliary results.

We first prove a generalized case to Theorem 2.5.2.

Lemma 4.4.1. *Let $(X^4, \partial X)$ be a complete Riemannian manifold with* $\mathrm{Ric}_2^X \geq 0$ *and $(M^3, \partial M)$ a FBMH in X. If either*

- *u is a smooth harmonic function on M with Neumann boundary condition and $\mathrm{II}_2^{\partial X} \geq 0$,*
- *or u is a smooth harmonic function on M with Dirichlet boundary condition and $H_{\partial X} \geq 0$,*

then

$$\int_M \phi^2 (\frac{1}{3}|\mathrm{II}|^2 |\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2) \leq \int_M |\nabla \phi|^2 |\nabla u|^2.$$

Proof. The proof of the Neumann case is in Theorem 2.5.2.

For Dirichlet case, the same proof of applies to get the following (without assuming any boundary condition),

$$\begin{aligned} &\int_M \phi^2 (\frac{1}{3}|\mathrm{II}|^2 |\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2) \\ \leq &\int_M |\nabla \phi|^2 |\nabla u|^2 + \int_{\partial M} \phi^2 (\frac{1}{2}\nabla_{\nu_{\partial M}} |\nabla u|^2 - \mathrm{II}_{\partial X}(\nu_M, \nu_M)|\nabla u|^2). \end{aligned} \tag{4.4.1}$$

We have $0 = \Delta_M u = \Delta_{\partial M} u + \nabla^2 u(\nu_{\partial M}, \nu_{\partial M}) + H_{\partial M} \nabla_{\nu_{\partial M}} u$. The Dirichlet boundary condition implies $\Delta_{\partial M} u = 0$ and $\nabla u = \nabla_{\nu_{\partial M}} u \cdot \nu_{\partial M}$,

$$\begin{aligned} \frac{1}{2}\nabla_{\nu_{\partial M}} |\nabla u|^2 &= \nabla_{\nu_{\partial M}} \nabla u \cdot \nabla u \\ &= (\nabla_{\nu_{\partial M}} u) \nabla^2 u(\nu_{\partial M}, \nu_{\partial M}) \\ &= -(\nabla_{\nu_{\partial M}} u)^2 H_{\partial M} = -H_{\partial M} |\nabla u|^2. \end{aligned}$$

Plug in the last computation into (4.4.1) and use $H_{\partial X} \geq 0$ then we obtain the desired inequality. □

Remark 4.4.2. Note that $\mathrm{II}_2^{\partial X} \geq 0$ implies $H_{\partial X} \geq 0$. If ∂M has more than one component and ∂M is 2-convex, the same result as in Lemma 4.4.1 holds if one assumes Dirichlet boundary or Neumann boundary conditions for the harmonic function on different components.

We first recall some definitions and lemmas about parabolicity. Details about this notion can be found in [42], [15] and [65]. For regularity reasons, we assume here that each end $E \subset (M \setminus K)$ for some compact set K, if has corner points (where $\partial_0 E := \partial M \cap E$ and $\partial_1 E := \overline{E} \cap K$ intersect), then the interior corner has small angles (no more than $\frac{\pi}{8}$). See section 2.4 for more details on this.

Definition 4.4.3 ([65], Definition 4.4)**.** A component $E \subset (M \setminus K)$ for some compact set $K \subset M$ of a Riemannian manifold M is non-parabolic if there is a non-constant positive harmonic function $f : E \to (0, 1]$, and $\partial_{\nu_{\partial M}} f|_{\partial_0 E} = 0, f|_{\partial_1 E} = 1$. Otherwise E is parabolic.

Lemma 4.4.4 ([65], Theorem 4.12)**.** *If E is non-parabolic, then there is a unique function $f : E \to (0, 1]$ with Neumann boundary condition on $\partial_0 E$ and Dirichlet on $\partial_1 E$, such that if $g : E \to (0, 1]$ is also a harmonic function with Neumann boundary condition on $\partial_0 E$ and Dirichlet boundary condition on $\partial_1 E$, then $f \leq g$. We call this the minimal barrier function over the non-parabolic component E. Furthermore, we can assume the minimal barrier function has finite Dirichlet energy.*

Lemma 4.4.5 ([65], Lemma 4.10)**.** *If $K \subset K'$ are two compact sets in M and $E \subset (M \setminus K)$ is a non-parabolic component, then there is a component $E' \subset (M \setminus K')$ and $E' \subset E$, such that E' is non-parabolic.*

Using a proof similar to Theorem 4.12 in [65] we have the following.

Lemma 4.4.6. *If $K \subset K'$ are two compact sets in M and $E' \subset (M \setminus K')$ is a non-parabolic component, then the component $E \subset (M \setminus K)$ that contains E' must be non-parabolic.*

Equivalently, if $K \subset K'$ are two compact sets in M, and $P \subset (M \setminus K)$ is parabolic, then each component of $(M \setminus K') \cap P$ must be parabolic.

Definition 4.4.7. We say a Riemannian manifold M is parabolic if there is a point $x \in M$ and a small geodesic ball $B_r(x)$ for some $r > 0$ such that the connected set $M \setminus B_r(x)$ is parabolic. Otherwise we say M is non-parabolic.

We say that M has at most one non-parabolic end, if for any compact set $K \subset M$, there is at most one non-parabolic component of $M \setminus K$. If M is non-parabolic, then M has exactly one non-parabolic end.

We note that this definition is independent of the choice of $x \in M$ and $r > 0$. Indeed, if the connected set $M \setminus B_{r_1}(x_1)$ is parabolic, then so is the connected set $M \setminus B_{r_2}(x_2)$. Because if $M \setminus B_{r_2}(x_2)$ is non-parabolic, take R large so that $B_{r_1}(x_1) \subset B_R(x_2)$, then $M \setminus B_R(x_2)$ must have a non-parabolic component by Lemma 4.4.5, which then again implies that the set $M \setminus B_{r_2}(x_2)$ is non-parabolic by Lemma 4.4.6.

We recall Theorem 5.3 in [65], that a stable FMBH in a 4-manifold with non-negative "2-intermediate Ricci curvature" and "2-convex" boundary can only have at most one non-parabolic end if its volume is infinite. We further prove each component of ∂M must be non-compact.

Theorem 4.4.8. *If* $(M^3, \partial M) \to (X^4, \partial X)$ *is a FBMH and* $\mathrm{Ric}_2^X \geq 0, \mathrm{II}_2^{\partial X} \geq 0$*, then*

- *either* $\mathrm{Vol}(M) < \infty$
- *or* $\mathrm{Vol}(M) = \infty$ *and* M *has at most 1 non-parabolic end.*

If M *is non-parabolic and* $\mathrm{Vol}(M) = \infty$*, then each component of* ∂M *must be non-compact.*

Proof. The proof that M with infinite volume must have at most 1 non-parabolic end is the same as Theorem 5.3 in [65]. We briefly summarize it here.

Given any compact set $K \subset M$, if $E_i (i \in \{1, 2\})$ is non-parabolic in $M \setminus K$, then on each E_i there is a minimal barrier function f_i with finite Dirichlet energy. Then using these two barrier functions one can construct a non-constant harmonic function on M with finite Dirichlet energy and Neumann boundary condition on ∂M. Using a linear cut-off function for ϕ in Lemma 4.4.1, we can show that this implies $\mathrm{Vol}(M)$ must be finite.

We now show that if M is non-parabolic, then either M has finite volume, or each component of ∂M must be non-compact.

Assume $\partial_1 M$ is a compact component of ∂M and denote $\partial_0 M := \partial M \setminus \partial_1 M$ (this could be the empty set).

We minimize Dirichlet energy on $B_R(x) \supset \partial_1 M$ for some $x \in M$ and $R > 0$, among functions f such that,

$$f|_{\partial_1 M} = 1, f|_{\partial' B_R(x)} = 0, \nabla_{\nu_{\partial M}} f|_{\partial_0 M} = 0.$$

As $R \to \infty$, the minimizer f_R converges in $C^2_{\text{loc}}(M)$ to a harmonic function f_∞, and $(f_\infty)|_{\partial_1 M} = 1, \nabla_{\nu_{\partial M}} f_\infty|_{\partial_0 M} = 0$.

Let g_∞ be the minimal barrier function on the non-parabolic end $E \subset (M \setminus B_r(x))$ for a fixed $r > 0$ so that $\partial_1 M \subset B_r(x)$. Then $f_R|_{E\cap B_R(x)} < g_\infty|_{E\cap B_R(x)}$, passing to the limit we get $f_\infty|_E \leq g_\infty|_E$, which implies f_∞ is non-constant.

The minimizing solution f_R has decreasing Dirichlet energy as $R \to \infty$. This implies that f_∞ is a non-constant harmonic function with finite energy and Dirichlet boundary condition on $\partial_1 M$ and Neumann boundary condition on $\partial_0 M$. By Remark 4.4.2 and Lemma 4.4.1 we get that $\text{Vol}(M)$ must be finite, a contradiction. □

We now prove each component of the boundary ∂M must has an end in the non-parabolic end of M.

Lemma 4.4.9. *Consider $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ a FBMH and $\text{Ric}_2^X \geq 0, \mathrm{II}_2^{\partial X} \geq 0$. Assume $(M, \partial M)$ is non-parabolic, $(C_k)_{k\in\mathbb{N}}$ is a sequence of compact sets and $E_k \subset (M \setminus C_k)$ is a sequence of nested non-parabolic components, if $\text{Vol}(M) = \infty$ then any connected component Γ of ∂M, must have $(\Gamma \setminus C_k) \cap E_k \neq \emptyset$ for any $k \in \mathbb{N}$.*

Proof. Assume Γ is a component of ∂M and there is $k_0 \in \mathbb{N}$ such that $(\Gamma \setminus C_{k_0}) \cap E_{k_0} = \emptyset$, then for any $k \geq k_0$, $(\Gamma \setminus C_k) \cap E_k = \emptyset$. Note that $\Gamma \setminus C_k \neq \emptyset$ since Γ is non-compact by Lemma 4.4.8.

We now minimize Dirichlet energy on $C_k (k \geq k_0)$ among functions satisfying the following,

$$f|_{\Gamma\cap C_k} = 1, \nabla_{\nu_{\partial M}} f|_{C_k \cap (\partial M \setminus \Gamma)} = 0, f|_{\partial' C_k \cap E_k} = 0, f|_{\partial' C_k \setminus E_k} = 1.$$

Denote the minimizer as $f_k : C_k \to [0, 1]$, and the minimal barrier function on E_{k_0} as f_0. Then using extension by constants, $\int_{C_k} |\nabla f_k|^2$ is non-increasing. By maximum principle, $f_k|_{E_{k_0}\cap C_k} \leq f_0|_{E_{k_0}\cap C_k}$, and passing to the limit we have f_k converges in $C^2_{\text{loc}}(M)$ to a harmonic function $f_\infty : M \to (0, 1]$ such that $f_\infty|_\Gamma = 1, \nabla_{\nu_{\partial M}} f_\infty|_{\partial M \setminus \Gamma} = 0$, and f_∞ is non-constant, because $f_\infty|_{E_{k_0}} \leq f_0|_{E_{k_0}}$.

In total we get a non-constant harmonic function with finite Dirichlet energy, using Remark 4.4.2 we get that the volume of M must be finite. □

4.5 Proof of almost linear volume growth and Rigidity

We can now start to prove Theorem 4.1.1. We can first pass to the universal cover of M, in this section, we assume that the FBMH M is simply connected. We assume all the assumptions of Theorem 4.1.1. By scaling we may assume that $H_{\partial X} \geq 2$.

We start with a lemma that shows that ∂M must be connected if M is simply connected.

Recall we use the notation ∂M to denote the boundary of a continuous manifold $(M, \partial M)$, and if Ω is an open subset of M, we denote the topological boundary as $\partial' \Omega$.

Lemma 4.5.1. *Assume $(M, \partial M)$ as in Theorem 4.1.1 is simply connected, each component of ∂M is non-compact (by Lemma 4.4.8), and each component of ∂M must have an end in the end $(E_k)_{k\in\mathbb{N}}$ in the sense of Lemma 4.4.9 (we do not assume (E_k) is non-parabolic), then ∂M is connected.*

Proof. Now let $\partial_1 M, \partial_2 M$ be two non-compact components. We may take the compact set $C_k := B_{3k\pi}(x)$ for some $x \in M$, and E_k is a nested sequence of sets in $M \setminus C_k$. Then using Corollary 4.3.3 for $H_0 = 2, d_0 = \pi, a_0 = 1$, we can find a θ-bubble $\Sigma_k = \partial' \Omega_k$ inside $C_{k+1} \setminus C_k$, with each component Σ_k^α of Σ_k having $|\partial \Sigma_k^\alpha| \leq 2\pi$ and $d_{\Sigma_k}(z, \partial \Sigma_k^\alpha) \leq 2$ for $z \in \Sigma_k^\alpha$.

By simply connectedness of M, $\partial' E_k$ must be connected for any $k \in \mathbb{N}$ and $(E_k \setminus E_{k+1}) \cap C_{k+1} =: M_k$ must also be connected (see [15], [65],[66]).

Now take a point $p_1 \in \partial_1 M$ and $p_2 \in \partial_2 M$, for k larger than some k_0 we may assume that $p_1, p_2 \in C_k$. Since both $\partial_1 M$ and $\partial_2 M$ are non-compact and has an end in $(E_k)_{k\geq k_0}$, we must have $\partial' E_k \cap \partial_i M \neq \emptyset$ for $k \geq k_0$ and $i \in \{1, 2\}$. That is, any path in $\partial_i M$ connecting p_i to some point $q_i \in E_k \cap \partial_i M$ must intersect $\partial' E_k \cap \partial_i M$. If $q_i \in \partial' E_{k+1}$, then a path $\overline{p_i q_i}$ in $\partial_i M$ must intersect $\partial \Sigma_k$. Since each component of Σ_k is a disk, we must have that $\overline{p_1 q_1}$ intersect $\partial \Sigma_k^\alpha$, and $\overline{p_2 q_2}$ intersect $\partial \Sigma_k^\beta$ for $\alpha \neq \beta$. We can now connect q_1, q_2 with a path in $\partial' E_{k+1}$, and connect p_1, p_2 with a path in C_{k_0}. This gives us a loop in M with non-trivial intersection to a disk, contradicting simply connectedness.

In total, we have shown that at most one component of ∂M can have an end in $(E_k)_{k\in\mathbb{N}}$, so ∂M is connected. $\square$

We now assume that $(E_k)_{k\in\mathbb{N}}$ is the non-parabolic end of M with respect to $C_k := B_{3k\pi}(x)$ for some $x \in \partial M$. If M is parabolic then $E_k = \emptyset$. Assume ∂M is connected. We denote $P_k := (M \setminus C_k) \setminus E_k$, and each component of P_k is parabolic. Denote $M_k := (E_k \setminus E_{k+1}) \cap C_{k+1}$. We now have the following decomposition of M,

$$\begin{aligned} M = B_{3\pi}(x) \cup E_1 \cup P_1 &= B_{3\pi}(x) \cup P_1 \cup (M_q \cup P_2 \cup E_2) \\ &= B_{3\pi}(x) \cup (\cup_{k=1}^m P_k) \cup (\cup_{k=1}^m M_k) \cup E_{k+1} \end{aligned}$$

Proof of Theorem 4.1.1. Using Corollary 4.3.3 we can find a θ-bubble $\Sigma_k = \partial'\Omega_k$ inside $C_{k+1} \setminus C_k$ with $|\partial\Sigma_k^\alpha| \leq 2\pi, d_{\Sigma_k}(z, \partial\Sigma_k^\alpha) \leq 2$ for any component Σ_k^α of Σ_k and $z \in \Sigma_k^\alpha$. In particular, we have the diameter of each Σ_k^α is at most $\pi + 4$.

Claim: there is a unique component Σ_k^α with $\partial\Sigma_k^\alpha \neq \emptyset$ separating $\partial' E_k$ and $\partial' E_{k+1}$, that is, any path from $\partial' E_k$ to $\partial' E_{k+1}$ must intersect Σ_k^α.

Take $q_{k+1} \in \partial' E_{k+1} \cap \partial M$, connect $x \in \partial M$ and q_{k+1} with a path in ∂M, then $\overline{xq_{k+1}}$ must intersect $\partial' E_k \cap \partial M$ at some point q_k, and must intersect some component Σ_k^α with $\partial\Sigma_k^\alpha \neq \emptyset$. Now if there is another path γ from $\partial' E_k$ to $\partial' E_{k+1}$ that is disjoint from Σ_k^α, as in Lemma 4.5.1, since $\partial' E_k$ is connected for all k, we can find a loop in M having non-trivial intersection with the disk Σ_k^α, a contradiction to simply connectedness. We finished the proof of the above claim.

Claim: $\sup_k \operatorname{diam}_M(M_k) \leq C$ for some constant $C > 0$.

Take $y, z \in M_{k+1}$ and take the geodesic line $\overline{xy}, \overline{xz}$. Now use $\overline{xy} \cap \partial E_k \neq \emptyset$ and $\overline{xy} \cap \partial E_{k+1} \neq \emptyset$ together with the first claim, we get that $\overline{xy} \cap \Sigma_k^\alpha \neq \emptyset$. Let $y_k \in \overline{xy} \cap \Sigma_k^\alpha$. Similarly, we get $\overline{xz} \cap \Sigma_k^\alpha \neq \emptyset$ and $z_k \in \overline{xz} \cap \Sigma_k^\alpha$. Now we have,

$$|\overline{yz}| \leq |\overline{yy_k}| + |\overline{y_k z_k}| + |\overline{z_k z}| \leq 6\pi + (\pi + 4) + 6\pi = C.$$

Using Lemma 2.3.1 and Lemma 2.3.3, we get that there is a constant $C_0 > 0$ such that $\sup_k \operatorname{Vol}(M_k) \leq C_0$.

We now use a suitable cut-off function to show the the non-parabolic end of M has linear volume growth.

Let φ_k be a cut-off function such that $\varphi_k|_{B_{3k\pi}}(x) = 1, \varphi_k|_{(M\setminus B_{6k\pi}(x))} = 0$ and with $|\nabla\varphi_k| \leq \frac{2}{3k\pi}$ everywhere and $\varphi_k|_{P_m}$ is constant for $k \leq m \leq 2k$.

On the parabolic ends P_m we use the existence of harmonic functions converging to 1 everywhere, with Dirichlet energy converging to 0 (see [42], [15], [65]). That is, let u_m^l be harmonic functions on P_m, with $u_m^l|_{\partial' P_m} = 1$, $u_m^l|_{P_m} \to 1$ as $l \to \infty$, and $\int_{P_m} |\nabla u_m^l|^2 \to 0$. We may further assume that, by choosing l large (depending on k, m), we have $\int_{P_m} |\nabla u_m^l|^2 \leq \frac{1}{k}2^{-m}$.

We define $v_k^l : M \to [0,1]$ such that $v_l^k|_{P_m} = u_m^l$ for $m \leq 2k$, and v_k^l is constant otherwise.

Let $\phi = \varphi_k v_k^l$, then recall the stability inequality of M,

$$
\begin{aligned}
&\int_M (\mathrm{Ric}_X(\nu_M, \nu_M) + |\mathrm{I\!I}_M|^2)\varphi_k^2(v_k^l)^2 + \int_{\partial M} \mathrm{I\!I}_{\partial X}(\nu_M, \nu_M)\varphi_k^2(v_k^l)^2 \\
&\leq 2\int_M |\nabla\varphi_k|^2(v_k^l)^2 + 2|\nabla v_k^2|^2\varphi_k^2 \\
&\leq 2\sum_{m=k}^{2k}(\frac{2}{3k\pi})^2 \int_{M_m} (v_k^l)^2 + 2\sum_{m=1}^{2k}\int_{P_m} |\nabla v_k^l|^2 \\
&\leq \sum_{m=k}^{2k} \frac{10}{k^2}\,\mathrm{Vol}(M_m) + 2\sum_{m=1}^{2k}\frac{1}{k}2^{-m} \\
&\leq (10C_0 + 4)\frac{1}{k} \to 0 \quad \text{as } k \to 0.
\end{aligned}
$$

Since $\varphi_k v_k^l \to 1$ everywhere, and all terms on the left hand side of the inequality is non-negative by assumption, we obtain the desired rigidity results. □

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