Capillary Hypersurfaces and Variational Methods

in Positively Curved Manifolds with Boundary

Yujie Wu

University of Potsdam

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Minimal Surfaces in Real Life

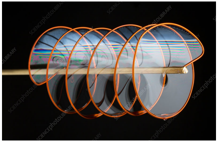
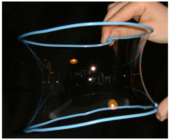
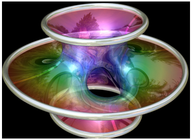




Image credit: Ted Kinsman, Blinking Spirit, Paul Nylander

Minimizing area while fixing the boundary (the wire): existence and regularity.
This is the Plateau's problem.



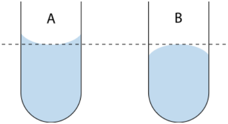
Image credit: Malte Sörensen, Kate Fraser, Joaquim Alves Gaspar

As one blows air into a soap bubble, the surface tension increase while enclosing a fixed "volume" inside the bubble.
Does the sphere minimize area given a fixed volume inside?
This is the isoperimetric problem.

Minimal Surfaces in Real Life

When we put liquid into a tube, the surface tension balances with adhesion between the tube and the liquid.

Capillary Action



A: Capillary attraction.
B: Capillary repulsion.

Image credit:Jleedev

No gravity: surfaces have constant mean curvature and constant angle along the container.

Applications of capillary action can be seen in many aspects of life.

Image Credit: Pat Hastings, Content Pixie

Minimal Surfaces for Mathematicians

- In 1762, Lagrange found the Euler-Lagrange equation for Plateau's problem of a graph $z = z(x, y)$ in $\mathbb{R}^3$,

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = 0.$$

- He found only one solution, the plane. A surface satisfying this equation is a critical point to the area functional, and is called a "minimal surface".
- In 18 and 19th century, more minimal surfaces are discovered, including the catenoid and helicoid (1744 Euler, 1776 Meusnier).
- The Plateau's problem for surfaces was completely solved in 1930 independently by Douglas and Radó.

Minimal Surfaces in Modern Days

- Extending the existence and smoothness of minimizers of Plateau's problem to higher dimensions turn out to be difficult.
- Singularities could occur for hypersurfaces in dimension 8 or higher, or for codimension 2 or more.
- Efforts in these directions contributed massively to the development of geometric measure theory.
- Meanwhile, minimal surfaces also find other geometric applications, one of which is the study of manifolds with positive scalar curvature (PSC), e.g. Geroch Conjecture.

Schoen-Yau 1979, Gromov-Lawson 1983, Geroch Conjecture

Consider X^n a closed manifold and $\mathbb{T}^n$ the n-torus ($3 \leq n \leq 7$), then $\mathbb{T}^n \# X$ has no PSC metric.

Schoen-Yau 1979, Positive Mass Theorem

Let (M^n, g) be an asymptotically flat manifold with $R_g \geq 0$, $3 \leq n \leq 7$, then its ADM mass $m_g \geq 0$, and $m_g = 0$ if and only if M is isometric to the Euclidean space.

Geroch Conjecture $\Rightarrow$ Positive Mass Theorem

Idea of Proof.

We show how the Geroch conjecture implies $m_g \geq 0$ in this setting.

- Lokhamp: if $m_g < 0$, then M has a metric $\hat{g}$ with $R_{\hat{g}} \geq 0$, $(M \setminus K, \hat{g})$ is isometric to $\mathbb{R}^n \setminus B_R(0)$, and $R_{\hat{g}}(x_0) > 0$.
- Cut $(M, \hat{g})$ with a large cube and identify the boundary so that now $(M, \hat{g}) \approx \mathbb{T}^n \# X^n$ for a closed manifold X^n.
- Kazdan-Warner, Kazdan: for a closed manifold $N^n, n \geq 3$ with $R_N \geq 0, \mathrm{Ric} \not\equiv 0$, then N has a PSC metric.
- Then we obtain a contradiction.

□

Proof of Geroch Conjecture

Geometric Idea

Stable minimal hypersurfaces of a PSC manifolds also admit a metric of PSC (after a conformal change of metric). This method is called Schoen-Yau's conformal descent method.

Proof $\mathbb{T}^n$ has no PSC metric

- If $n = 2$ the claim follows from Gauss-Bonnet.
- We now induct using the method of "Conformal Descent".
- A torus $\mathbb{T}^n$ has enough topology so we can minimize in a non-trivial homology class inductively, to obtain a chain of nested stable minimal hypersurfaces, $\mathbb{T}^n \supset \Sigma_{n-1} \supset \Sigma_{n-2}, ..., \supset \Sigma_2$.
- For $n \leq 7$, these minimizers must be smooth.
- Now Σ_2 has PSC and must be spheres (Gauss-Bonnet).
- Σ_2 has non-trivial H^1 by induction. A contradiction.

The Method of μ-bubble

Stable minimal hypersurfaces do not always exists (as we will prove later in some cases). How to generalize?
We can trade minimality for existence and stability inequality.

Definition (Gromov 1996, 2018)

Roughly speaking, a μ-bubble in a manifold (M^n, g) is a smooth open set Ω that minimizes the following functional, given $h \in C^\infty(M)$,

$$\mathcal{A}(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h d\mathcal{H}^n.$$

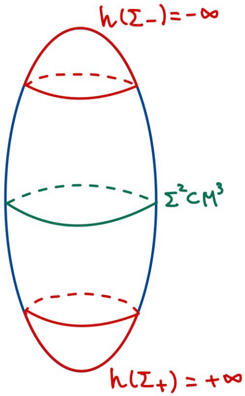
- If $h = 0$ then this is the area functional.
- If $n \leq 7$ then a minimizer $\Sigma = \partial\Omega$ is always smooth.

The first variation: $H_\Sigma = h|_\Sigma$. The $\mathcal{A}(\cdot)$ is also called the "prescribed mean curvature" (PMC) functional.

The Method of μ-Bubble: A Model Case for 3-Manifolds

We choose h on (M^3, g) so that the sets $\Sigma_\pm := \{x \in M, h(x) = \pm\infty\}$ serves as "barriers" to constrain and make sure a minimizer must exist.

- A smooth minimizer must contain Σ_+ and be disjoint from Σ_-.
- The second variation is non-negative (stability inequality).
- $R_g \geq 2$, and $d_0 := \mathrm{diam}(M) > 2\pi$: $\implies$ Σ also admit PSC.
- A PSC surface has bounded diameter.
- Localization if M is non-compact.



Topology and Geometry of PSC manifolds

The μ-bubble method allows us to obtain new geometric estimates for PSC 3-manifolds.

- Topologically, a 3-manifold with uniform PSC must be connected sum of $\mathbb{S}^2 \times \mathbb{S}^1$ and space forms (quotients of $\mathbb{S}^3$).
  - Uses Gromov-Lawson 1983, Geometrization proved by Perelman in 2003, and combined works of Chang-Weinberger-Yu 2010, Besseeres-Besson-Maillot 2011, Wang (using μ-bubbles) 2023.
- More Quantitatively: a complete 3-manifold (M, g) with $R_g \geq R_0 > 0$ is close to being "one-dimensional".
  - Liokumovich-Maximo 2020, Liokumovich-Wang 2023
  - There is a continuous map $f : M \to \mathbb{R}$ such that every component of a fiber must have bounded diameter and area.

Urysohn Width and the μ-Bubble Method

Definition

A metric space (X, d) has k-Urysohn width bounded by $d_0 > 0$, if there is a continuous map to a k-dimensional space $f : X \to G^k$, such that $\mathrm{diam}_d f^{-1}(g) \leq d_0$ for all $g \in G$.

Example

Any compact n-manifold has bounded 0-Urysohn width.
Positive Ricci lower bound gives uniform bound on 0-Urysohn width.
What about scalar curvature?

Conjecture (Gromov, 2017)

If (X^n, g) with $n \geq 2$ is a closed Riemannian manifold with $R_g \geq 1$, then its $(n-2)$-Urysohn width is bounded by $c(n) > 0$.

Urysohn Width and the μ-Bubble Method

The μ-bubble method can be used to give a short proof of the simply connected case of Gromov's conjecture.

Theorem (Chodosh-Li, 2024)

If (M, g) is a simply connected 3-manifold with $R_g \geq 2$, then the 1-Urysohn width of M is bounded from above by 10π.

Proof.

Fix a point $x \in M$, and consider the bands $M_k := B^M_{2(k+1)\pi}(x) \setminus B^M_{2k\pi}(x)$, since $R_g \geq 2$ over each band of length at least 2π, we can

- put a μ-bubble called Σ^2 inside with diameter no more than 2π;
- using simply connectedness we know that Σ is separating in M_k;
- using triangle inequality we get that the diameter of each M_k is no more than 10π.

□

Applications of 1-Urysohn Width Bound

Theorem (Chodosh-Li, 2024)

The following two generalized Geroch Conjecture holds,

- *Closed aspherical 4 or 5 manifolds has no PSC.*
- $\mathbb{T}^n \# X (2 \leq n \leq 7)$ *for any manifold* X *has no complete PSC metric.*

Remark

A torus is aspherical. The extensions follows the idea of the proof of Schoen-Yau but generalized in the sense that here we need to find (generalized) minimal surfaces in a space with little topology.

Rigidity of Stable Minimal Hypersurfaces

Earlier rigidity results using stability of $M^n \subset X^{n+1}$:

$$\int_M |\nabla\phi|^2 \geq \int_M (\mathrm{Ric}_X(\nu_M, \nu_M) + |\mathrm{I\!I}_M|^2)\phi^2.$$

- If $\mathrm{Ric}_X \geq 0$, then any compact stable minimal hypersurface is totally geodesic, and $\mathrm{Ric}_X(\nu_M, \nu_M) = 0$ along M^n (Simons 1968).
- $\mathrm{Ric}_X > 0$ implies non-existence.
- If $R_X \geq 1$ then M admit a metric of PSC (Scheon and Yau, 1979).
- If $n = 2$ and M is complete non-compact, then $R_X \geq 0$ implies M must be conformal to a plane or a cylinder. In the latter case, M must be totally geodesic, intrinsically flat, (Fischer-Colbrie and Schoen 1980).

What is the correct assumptions when $n = 3$ and M is non-compact?

- There exists a stable totally geodesic $\mathbb{R}^3$ embedded in $(\mathbb{R}^4, g)$ with $\sec > 0$. So $\sec > 0$ does not imply non-existence.
- $\mathrm{Ric}_X \geq 1$ also cannot rule out existence using the method of second variation.

Rigidity of Stable Minimal Hypersurfaces

Curvature hierarchy of a manifold X:

- $\sec \geq 0 \implies \mathrm{Ric}_2 \geq 0 \implies \mathrm{Ric} \geq 0$.

Theorem (Chodosh-Li-Stryker, 2022)

Consider (X^4, g) has weakly bounded geometry and

$$Ric_2^X \geq 0, \quad R_X \geq R_0 > 0.$$

Then any complete two-sided stable minimal hypersurface $M^3 \hookrightarrow X^4$ must have

$$|\mathrm{II}_M| = 0, \quad Ric(\nu_M, \nu_M) = 0,$$

for ν_M a choice of unit normal along M.

In particular, $\mathbb{S}^4$ has no complete two-sided stable minimal hypersurfaces.

Ingredients of the Proof.

We may assume M is simply connected. Recall stability:

$$\int_M |\nabla\phi|^2 \geq \int_M (\mathrm{Ric}_X(\nu_M,\nu_M) + |\mathrm{II}_M|^2)\phi^2.$$

- Goal: show M has almost linear volume growth.
- X^4 is PSC $\Rightarrow$ M^3 inherits PSC $\Rightarrow$ M has bounded 1-Urysohn width.
- We still need to control the number of ends of M, this relates to the notion of parabolicity.
- If M is parabolic, then on each end of M, one can find a sequence of harmonic functions u_i that are good test functions for stability,

$$\int_M |\nabla u_i|^2 \to 0, \quad u_i \xrightarrow{C^\infty_{\mathrm{loc}}(M)} 1.$$

- If not, $\mathrm{Ric}_2^X \geq 0$ implies a Liouville theorem: harmonic function on M with finite energy must be constant.
- This allows us to show M has at most one non-parabolic end.

□

Free Boundary Minimal Hypersurfaces

Definition

An free boundary minimal hypersurface $(M^n, \partial M) \hookrightarrow (X^{n+1}, \partial X)$ is a critical point to the area functional among all variations that send ∂X to ∂X, we call M a FBMH.

Equivalently, this means $H_M = 0$, and M meets with ∂X orthogonally.

Theorem (W., 2023)

Consider $(X^4, \partial X, g)$ has weakly bounded geometry

$$Ric_2^X \geq 0, R_X \geq R_0 > 0, \mathrm{II}_{\partial X} \geq 0.$$

Then any complete two-sided stable free boundary minimal hypersurface $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ must have

$$\mathrm{II}_M = 0, Ric_X(\nu_M, \nu_M) = 0, \mathrm{II}_{\partial X}(\nu_M, \nu_M) = 0$$

- Hierarchy of convexity: $\mathrm{II}_{\partial X} \geq 0 \implies \mathrm{II}_2^{\partial X} \geq 0 \implies H_{\partial X} \geq 0$.
- Rearranged stability inequality, $\mathrm{Ric}_2^X \geq 0$ and $\mathrm{II}_2^{\partial X} \geq 0$ (2-convexity of the boundary) $\implies$ the same Liouville theorem holds for M.
- Using free boundary μ-bubbles and $H_{\partial X} \geq 0$ we can show the 1-Urysohn width bound also holds for M.

Trading Uniform PSC with Uniform Mean Convexity

The unit ball $\mathbb{B}^4$ does not have PSC, but has uniformly convex boundary. Can we trade the $R_g \geq 1$ with $H_{\partial X} \geq 1$?

- Franz 2022 proved, if $(X^3, \partial X)$ has "weakly positive geometry",
  - either $R_X \geq R_0 > 0, H_{\partial X} \geq 0$ and ∂X has no minimal component,
  - or $R_X \geq 0, H_{\partial X} \geq H_0 > 0$,

  then any stable FBMH $M \hookrightarrow X$ must be a compact disc with intrinsic diameter bounded by a constant $C(R_0, H_0)$.

Theorem (W,. 2025)

Consider a 4-manifold $(X^4, \partial X)$ with weakly bounded geometry, assume

$$Ric_2^X \geq 0, \mathrm{II}_{\partial X} \geq 0, H_{\partial X} \geq H_0 > 0.$$

If $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ is a complete stable FBMH, then M has

$$|\mathrm{II}_M| = 0, Ric_X(\nu_M, \nu_M) = 0, \mathrm{II}_{\partial X}(\nu_M, \nu_M) = 0.$$

The assumption of PSC allows us to use μ-bubbles, a key tool to obtain geometric control. What should we do now?

Capillary Hypersurfaces

Capillary surfaces help us study manifolds with non-negative scalar curvature (NNSC) and uniformly mean convex boundary.

Definition

A capillary hypersurface $\Sigma^n = \partial\Omega$ in M^{n+1} is a critical point to,

$$E_c(\Omega) := \text{Area}(\partial\Omega) - \cos\theta \text{Area}(\overline{\Omega} \cap \partial M),$$

among variations that fix the volume ratio $\lambda_0 := \frac{\text{Vol}(\Omega)}{\text{Vol}(M)}$.

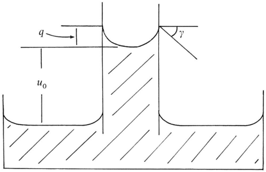


Figure: Robert Finn

Equivalently, Σ is a capillary hypersurface if it has constant mean curvature and intersect with ∂M at a constant angle.

Generalized Capillary Hypersurfaces: θ-Bubbles

The idea: having NNSC and Mean Convex Boundary can be also inherited by (generalized) capillary hypersurfaces.

Definition

Consider a manifold with boundary $(M^{n+1}, \partial M)$, given a smooth function $\theta \in C^\infty(\partial M)$, a θ-bubble $\Sigma = \partial\Omega$ is a minimizer to,

$$E_\theta(\Omega) := \mathrm{Area}(\partial\Omega) - \int_{\overline{\Omega}\cap\partial M} \cos\theta.$$

First and Second Variation

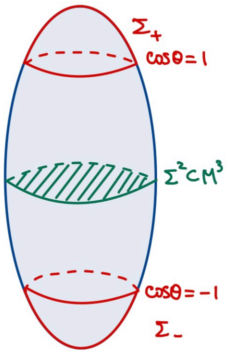
$$H_\Sigma = 0, \quad \langle \nu, \bar{\nu} \rangle = \cos\theta(x)$$

We may call a θ-bubble, a "prescribed contact angle" surface.

The Method of θ-Bubble: A Model Case for 3-Manifolds

We choose θ on ∂M so that the sets $\Sigma_\pm := \{x \in \partial M, \cos\theta = \pm 1\}$ serves as "barriers" to constrain and make sure a minimizer Σ must exist.

- Solomon-White: if $H_{\partial M} \geq 0$ and $\cos\theta \equiv 1$, then Σ must be minimizing across ∂M, either disjoint to ∂M, or equal to a connected component of ∂M.
- Using a similar argument, here $H_{\partial M} > 0$ gives a minimizer always exists and $\partial\Sigma \subset \{|\cos\theta| < 1\}$, Σ is smooth if $\dim(\Sigma) \leq 4$.
- Stability inequality: $R_M \geq 0, H_{\partial M} \geq 2$ and $d_0 := \mathrm{diam}(\partial M) > \pi$, $\implies R_\Sigma \geq 0, H_{\partial\Sigma} > 0$.
- This leads to localization of $\partial\Sigma$ when M is non-compact. Further estimates $d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}$ localizes Σ totally.



The Method of θ-bubble

Using θ-bubbles, we can obtain the following geometric estimates.

Theorem (W., 2024)

If $(S^2, \partial S)$ is a complete connected manifold with $R_S \geq 0$ and $k_{\partial S} \geq 1$, then S is a compact topological disk with $|\partial S| \leq 2\pi$ and $d(x, \partial S) \leq 1$ for any $x \in S$. Furthermore, if $|\partial S| = 2\pi$ then S is isometric to the unit disk in $\mathbb{R}^2$.

Theorem (W., 2024, Obstruction to Gromov's Fill-In Question)

If $(M^3, \partial M)$ is a complete simply connected Riemannian manifold with $R_M \geq 0, H_{\partial M} \geq 3$, then the 1-Urysohn width of ∂M with respect to the induced metric is at most 3π.

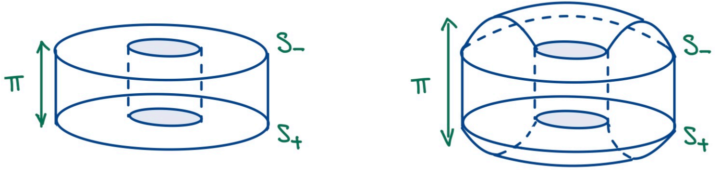
The Method of θ-Bubble

Theorem (Gromov 2020, Bandwidth Estimate)

Let $2 \leq n \leq 6$, consider $M = (\mathbb{T}^n \times [-1,1], g)$ such that $R_g \geq n(n+1)$, then $d_g(\mathbb{T}^n \times \{+1\}, \mathbb{T}^n \times \{-1\}) \leq \frac{2\pi}{n+1}$. And the bound is sharp.

Theorem (W., 2024, Bandwidth Estimate)

Let $(M^3, \partial M, g) = \Sigma_0 \times [-1,1]$ with $(\Sigma_0, \partial\Sigma_0)$ an orientable surface with $\chi(\Sigma_0) \leq 0$. If $R_M \geq 0, H_{\partial_0 M} \geq 2$ and $H_{\partial M} > 0$, then $d_{\partial M}(\partial S_+, \partial S_-) \leq \pi$, in particular $d_M(S_+, S_-) \leq \pi$.



The Method of θ-bubble

The idea is that we can chop ∂M into chunks of bounded diameter.

Corollary (W., 2024, Linear Growth of ∂M)

If $(M^3, \partial M)$ be a complete simply connected NNSC Riemannian manifold. If ∂M is uniformly mean convex and has weakly bounded geometry, then each end of ∂M has linear volume growth. In particular, if ∂M has finitely many ends, then ∂M has linear volume growth.

Remark

Linear volume growth in the interior can not be obtained for NNSC manifolds.

Back to Rigidity of FBMH in $\mathbb{B}^4$: Trading Mean Convexity for PSC

So far we are using that $(M^3, \partial M)$ inherits the NNSC and mean convexity through stability, and are only able to obtain control of ∂M.
Note M may have compact or disconnnected ∂M.
We need to further exploit stability to control the interior of M.

Rigidity of complete FBMH in $\mathbb{B}^4$

Theorem (W,. 2025)

Consider a 4-manifold $(X^4, \partial X)$ *with weakly bounded geometry, assume*

$$Ric_2^X \geq 0, \mathrm{II}_{\partial X} \geq 0, H_{\partial X} \geq H_0 > 0.$$

If $(M^3, \partial M) \hookrightarrow (X^4, \partial X)$ *is a complete stable FBMH, then* M *has*

$$|\mathrm{II}_M| = 0, Ric_X(\nu_M, \nu_M) = 0, \mathrm{II}_{\partial X}(\nu_M, \nu_M) = 0.$$

Ingredients of the Proof

The goal is still to show M has almost linear growth on an end.

- Rearranged stability inequality, $\mathrm{Ric}_2^X \geq 0$ and $\mathrm{II}_2^{\partial X} \geq 0$ together implies the same Liouville theorem holds for M.
- Further exploiting the Liouville theorem:
  - M has at most one non-parabolic end;
  - ∂M cannot have any compact component;
  - each component of ∂M must has an end in the only non-parabolic end M has.
- Now using simply-connectedness, we can exhaust the non-parabolic end of M using θ-bubbles and obtain this end has linear growth.

Thank You For Listening!

Jean Siméon Chardin, 1733-34

Marie Gale, 2012